

Differentiability of Spectral Functions  
for Symmetric Markov Processes

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### **Abstract**

We prove a large deviation principle for additive functionals corresponding to Kato measures by using the Gärtner-Ellis theorem. To this end, we prove the differentiability of spectral functions for symmetric Markov processes, in particular, Brownian motion, symmetric  $\alpha$ -stable process and relativistic  $\alpha$ -stable process.

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# 1 Introduction

As a useful approach in proving the large deviation principle, the Gärtner-Ellis theorem is well known. The Gärtner-Ellis theorem generalizes Cramér's method for the sum of independent identically distributed random variables. The objective of this thesis is to prove the large deviation principle for additive functionals of a symmetric Markov process, in particular, of a symmetric Lévy process by employing the Gärtner-Ellis theorem. For applying the Gärtner-Ellis theorem, we need to check two conditions:

- (I) the existence of the *logarithmic moment generating function* (LMGF) and its identification,
- (II) the “essentially smoothness” of the LMGF.

For the sake of introduction, we restrict our attention to the symmetric  $\alpha$ -stable process and explain how to check these conditions, although the symmetric  $\alpha$ -stable process can be replaced by more general symmetric Markov processes.

Let  $(P_x, X_t)$  be a symmetric  $\alpha$ -stable process on  $\mathbb{R}^d$  ( $0 < \alpha < 2$ ), the pure jump process generated by  $\mathcal{H} = -\frac{1}{2}(-\Delta)^{\alpha/2}$ . Let  $\mu$  be a *Green-tight* measure in the Kato class (in notation,  $\mu \in \mathcal{K}_d^\infty$ ) and  $A_t^\mu$  the positive continuous additive functional (PCAF) in the Revuz correspondence to  $\mu$  (for the definition of  $\mathcal{K}_d^\infty$ , see Definition 3.1 below). Let us denote by  $-C(\lambda)$  the bottom of the spectrum of  $\mathcal{H}^{\lambda\mu} = -\mathcal{H} - \lambda\mu$ :

$$-C(\lambda) = \inf \left\{ \mathcal{E}(u, u) - \lambda \int_{\mathbb{R}^d} u^2 d\mu : u \in C_0^\infty(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 dx = 1 \right\}, \quad \lambda \in \mathbb{R}^1.$$

Here  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is the Dirichlet form generated by the symmetric  $\alpha$ -stable process and  $C_0^\infty(\mathbb{R}^d)$  is the set of smooth functions with compact support. Then our main aim is to establish the large deviation principle for  $A_t^\mu$ :

**Theorem 1.1.** *Assume that  $d \leq 2\alpha$ . Then for  $\mu \in \mathcal{K}_d^\infty$ ,  $A_t^\mu/t$  obeys the large deviation principle with rate function  $I(\theta)$ :*

(i) *For any closed set  $F \subset \mathbb{R}^1$ ,*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P_x \left( \frac{A_t^\mu}{t} \in F \right) \leq - \inf_{\theta \in F} I(\theta).$$

(ii) *For any open set  $G \subset \mathbb{R}^1$ ,*

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log P_x \left( \frac{A_t^\mu}{t} \in G \right) \geq - \inf_{\theta \in G} I(\theta),$$

where  $I(\theta)$  be the Legendre transform of  $C(\lambda)$ :

$$I(\theta) = \sup_{\lambda \in \mathbb{R}^1} \{\theta\lambda - C(\lambda)\}, \quad \theta \in \mathbb{R}^1.$$

We first need to show the condition (I), the existence of the LMGF, that is, the existence of the limit,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log E_x[\exp(\lambda A_t^\mu)], \quad \lambda \in \mathbb{R}^1. \quad (1)$$

It was shown in [45] that in the case  $\alpha = 2$ , that is, the case of Brownian motion, the limit (1) exists for any Kato measure  $\mu \in \mathcal{K}_d$ . Moreover, if the Lévy measure  $J(dx)$  of a symmetric Lévy process is exponentially localized, that is, there exists a positive constant  $\delta$  such that

$$\int_{|x|>1} e^{\delta|x|} J(dx) < \infty, \quad (2)$$

then the limit (1) exists for any Kato measure associated with the Lévy process. For example, the Lévy measure of the relativistic  $\alpha$ -stable process is known to be exponentially localized ([8]). Situation is different for the symmetric  $\alpha$ -stable process which does not satisfy exponential localization, hence the method fails to work. Takeda [50] recently proved that the LMGF for the symmetric  $\alpha$ -stable process exists if  $\mu$  belongs to the restricted class  $\mathcal{K}_d^\infty \subset \mathcal{K}_d$ . Furthermore, he showed that the LMGF is identical to  $C(\lambda)$ . His method in [50] is completely different; he uses an ergodic theorem due to Fukushima [18]. In this way, the problem (I) is settled for additive functionals in Revuz correspondence to a measure in the Green-tight class. Hence to obtain the Theorem 1.1 by applying the Gärtner-Ellis theorem, it is enough to check the “essentially smoothness” of  $C(\lambda)$ . Therefore we concentrate on the proof of the differentiability of the spectral function  $C(\lambda)$  for  $\mu \in \mathcal{K}_d^\infty$ . Indeed the differentiability implies the essentially smoothness of the LMGF. We now sketch the proof of the differentiability.

When we study the differentiability of the spectral function  $C(\lambda)$  for symmetric  $\alpha$ -stable processes, we have to extend a critical theory to Schrödinger type operators with non-local principal part. We define a real number  $\lambda^+$  as follows:

$$\lambda^+ = \inf\{\lambda > 0 : C(\lambda) > 0\}.$$

We can apply the analytic perturbation theory [24] to prove the differentiability for  $\lambda > \lambda^+$  and we know that  $C(\lambda) = 0$  for  $\lambda < \lambda^+$ . Hence the main problem is the differentiability of  $C(\lambda)$  at  $\lambda = \lambda^+$ . For one or two-dimensional Brownian motion, Takeda [49] proved the differentiability at  $\lambda = \lambda^+$  (in this case,  $\lambda^+$  equals 0). For the proof of the differentiability, a well-known property of the one or two-dimensional Brownian motion, the null Harris recurrence, is used crucially. However, when  $d$  is greater than  $\alpha$ , the symmetric  $\alpha$ -stable process is known to be transient and the method in [49] is not directly applicable to transient  $\alpha$ -stable processes,  $d > \alpha$ . Nevertheless, if  $d \leq 2\alpha$ , we can use the method in [49] through, so-called, Doob’s  $h$ -transform. This is our key idea. To this end, we first note that Schrödinger type operators  $\mathcal{H}^{\lambda^+\mu} = -\mathcal{H} - \lambda^+\mu$  are critical. Here, the Schrödinger type operator  $\mathcal{H}^\mu$  is said to be *critical* if it does not possess the minimal Green function but possesses the harmonic function. In this sense, the criticality is regarded as an extended notion of recurrence property for Schrödinger semigroups. Indeed, a semigroup generated by the critical Schrödinger type operator can be transformed to a recurrent Markovian semigroup through the  $h$ -transform. More precisely, let  $P_t^{\lambda^+\mu}$  be the semigroup generated

by  $\mathcal{H}^{\lambda^+ \mu}$  and  $h$  a harmonic function of  $\mathcal{H}^{\lambda^+ \mu}$ . The Doob's  $h$ -transform is defined by

$$P_t^{\lambda^+ \mu, h} f(x) = \frac{1}{h(x)} P_t^{\lambda^+ \mu} (h(x) f(x)).$$

The  $h$ -transformed semigroup  $P_t^{\lambda^+ \mu, h}$  then becomes an  $h^2 m$ -symmetric Markovian semigroup, where  $m$  is the Lebesgue measure on  $\mathbb{R}^d$ . Since the existence of the Green function of  $P_t^{\lambda^+ \mu, h}$  is equivalent to that of  $P_t^{\lambda^+ \mu}$ , we can construct a recurrent  $h^2 m$ -symmetric Markov process with the semigroup  $P_t^{\lambda^+ \mu, h}$ . To apply the method in [49], we need study the following two properties for the  $h$ -transformed process: (a) Harris recurrence; (b) null recurrence.

To prove the property (a), we need to show that  $h$  is continuous. To obtain the continuity of  $h$ , we develop various methods depending on underlying symmetric Markov processes. For the transient Brownian motion [51], using the local property and the strong Markov property of Brownian motions we first construct a sequence of bounded continuous harmonic functions  $\{h_n\}$  on a open ball. We then obtain the continuous harmonic function on  $\mathbb{R}^d$  as a limit of  $h_n$  through the Ascoli-Arzelà theorem. To apply the Ascoli-Arzelà theorem, we use the Harnack inequality for Schrödinger operator with Kato potentials given by [6]. We would like to emphasize that this method is based on the ellipticity of the Laplacian. For the stable process we can not apply this method and thus need a different method. We first show the following equation for the harmonic function  $h$  by using the strong Markov property:

$$h(x) = E_x[h(X_{\tau_D})] + \lambda^+ E_x \left[ \int_0^{\tau_D} h(X_s) dA_s^\mu \right]. \quad (3)$$

The first term of the right hand side in the equation (3) is harmonic with respect to the principal part  $-\mathcal{H}$ . We show the continuity of the first term by using the Harnack inequality for non-local type operator given by Bass-Levin [4]. Concerning the second term of the right hand side in the equation (3) in the case of symmetric  $\alpha$ -stable processes, we show the continuity by the explicit expression of the Green function and the definition of the Kato class. We can not use this method for relativistic  $\alpha$ -stable processes since the explicit form of Green function is unknown. Thus we prove the continuity by a different method. We use the strong Feller property of  $P_t^{\lambda^+ \mu}$ ; if  $\mu \in \mathcal{K}_d^\infty$ , the Schrödinger semigroup  $P_t^\mu$  transforms the set of bounded Borel functions to the set of the bounded continuous function (it is called strong Feller property). Since the  $h$ -transformed process is recurrent, it is conservative, hence  $P_t^{\lambda^+ \mu, h} 1 = 1$ . By the definition of the  $h$ -transform, we know that  $P_t^{\lambda^+ \mu} h = h$ . Since we can show that the harmonic function  $h$  is bounded if  $\mu$  is the Green-tight Kato measure, we obtain the continuity of  $h$ . We would like to emphasize that this method is applicable to various symmetric Markov processes whose Green functions are explicitly unknown.

To show the strong Feller property of the semigroup, we need to consider various equivalence of definitions of the Kato class. We use probabilistic tools (Markov property, etc.) to prove the strong Feller property. In this paper, we define the Kato class by the analytical definition in terms of the resolvent (see Definition 3.1 below). For Brownian

motions and symmetric  $\alpha$ -stable processes, it is known that the analytical definition of Kato class is equivalent to the probabilistic one ([1], [56]). In [54], we confirm the equivalence for relativistic  $\alpha$ -stable processes by using arguments in [25] and [56]. Therefore we can check the property (a).

Next on the property (b), the null-recurrence is defined by the infinity of the total mass of invariant measure. Since the invariant measure of the  $h$ -transformed process is  $h^2m$ , the null recurrence of it is equivalent to  $h \notin L^2(m)$ . To this end, we need to study the asymptotic behavior of  $h$  at infinity. We can show by using the Harnack inequality that the asymptotic behavior of  $h$  is same to that of Green function, that is,

$$cG(x, 0) \leq h(x) \leq CG(x, 0), \text{ for } |x| > 1$$

where  $G(x, y)$  is the Green function for the underlying symmetric Markov process and  $c, C$  are positive constants. Since the explicit form of the function  $G(x, y)$  is known for the Brownian motion and the symmetric  $\alpha$ -stable process, we find that  $h \notin L^2(m)$  if and only if  $d \leq 2\alpha$  for the symmetric  $\alpha$ -stable process. In this way, we can check properties (b). As a result, we establish the Theorem 1.1. Recently, Rao, Song and Vondraček [32] obtained asymptotic behaviors of Green functions of Lévy processes including relativistic  $\alpha$ -stable processes. By using their result, we know that for the relativistic  $\alpha$ -stable process, its harmonic function does not belong to  $L^2(m)$  if and only if  $d \leq 4$  and prove the differentiability of its spectral function.

We now give the outline of this paper. In section 2, we prepare basic notions and assumptions related to Markov processes, such as Hunt processes, positive continuous additive functionals (PCAF), Dirichlet forms and convolution semigroup. We also introduce the notion of subordinators. In section 3, we define some classes of measures, such as Kato class  $\mathcal{K}_d$  and its subclass  $\mathcal{K}_d^\infty$  and  $\mathcal{S}_\infty$ . We introduce the Revuz correspondence, the correspondence between PCAF's and measures. In section 4, we define the spectral function and study its property. In section 5, we construct a finely continuous bounded harmonic function with respect to a critical Schrödinger type operator. In section 6, we prove the differentiability of the spectral function. To this end, we extend the Oshima's inequality to a critical Schrödinger type operator. We then prove the differentiability of spectral function when the Schrödinger type operator is null critical. Finally, we make a remark that the spectral function is not differentiable when the Schrödinger type operator is positive critical. In section 7, using the Gärtner-Ellis theorem, we prove the large deviation principle for additive functionals. In section 8, we prove the continuity of the harmonic function in various cases. In particular, we discuss Brownian motions, symmetric  $\alpha$ -stable processes and relativistic  $\alpha$ -stable processes. Finally in section 9, we give an example which shows that the condition (I) and (II) for the Gärtner-Ellis theorem, though sufficient, not necessary for the large deviation principle.

Throughout this paper,  $m$  is the Lebesgue measure and  $B(R)$  is an open ball with radius  $R$  centered at the origin. We use  $c, C, \dots, etc$  as positive constants which may be different at different occurrences.



## 2 Preliminaries

Let  $S$  be a locally compact separable metric space and  $m$  is a Radon measure with  $\text{supp}[m] = S$ . We now consider a  $\sigma$ -finite measure space  $(S, \mathcal{B}, m)$  and take as a real Hilbert space  $H$  the  $L^2$ -space  $L^2(S; m)$  consisting of square integrable  $m$ -measurable extended real valued functions on  $S$ . Let  $\mathbf{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, P_x, X_t)$  be a symmetric Hunt process on  $S$  and  $E_x$  be the expectation with respect to  $P_x$ . Here  $\{\mathcal{F}_t\}_{t \geq 0}$  is the minimal (augmented) admissible filtration and  $\theta_t, t \geq 0$ , is the shift operators satisfying  $X_s(\theta_t) = X_{s+t}$  identically for  $s, t \geq 0$ . Let  $P_t$  be the Markov semigroup of  $\mathbf{M}$ , that is for a suitable function  $f$ ,

$$P_t f(x) = E_x[f(X_t)].$$

Here we explain about Hunt processes. Let  $\mathbf{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, P_x, X_t)$  be as above. Let  $S_\Delta$  be one-point compactification of  $S$ . First we note following conditions.

- (i) for each  $x \in S_\Delta$  a stochastic process with state space  $S_\Delta$  is given by  $(\Omega, \mathcal{F}, P_x, (X_t)_{t \in [0, \infty]})$ ;
- (ii)  $P_x(X_t \in B)$  is  $\mathcal{B}$ -measurable in  $x \in S$  for each  $B \in \mathcal{B}$  and each  $t \geq 0$ ;
- (iii) (Markov property)

$$P_x(X_{t+s} \in B | \mathcal{F}_t) = P_{X_t}(X_s \in B), \quad P_x - a.s.$$

holds for all  $x \in S, B \in \mathcal{B}$  and  $t, s \geq 0$ ;

(iv) it holds  $P_\Delta(X_t = \Delta) = 1$ ; for all  $t \geq 0$ ;

(v)  $P_x(X_0 = x) = 1$  for all  $x \in S$ ;

Conditions (i)-(v) is satisfied, then the process  $\mathbf{M}$  is called a *normal Markov process*.

Let us consider a normal Markov process  $\mathbf{M}$  on  $(S, \mathcal{B})$  and assume the following additional conditions concerning the pair  $(\Omega, X_t)$ :

Condition **(H)**:

(a)  $X_\infty(\omega) = \Delta$  for all  $\omega \in \Omega$ ;

(b) with  $\zeta(\omega) := \inf\{t \geq 0 : X_t(\omega) = \Delta\}$  it follows that

$$X_t(\omega) = \Delta$$

for all  $t \geq \zeta(\omega)$ ;

(c) there exists a family of shift operators  $(\theta_t)_{t \in [0, \infty]}$  from  $\Omega$  to  $\Omega$  such that

$$X_{s+t}(\omega) = X_s(\theta_t \omega)$$

holds for all  $s, t \geq 0$ ;

(d) for each  $\omega \in \Omega$ , the sample path  $t \mapsto X_t(\omega)$  is right continuous on  $[0, \infty)$  and the left limit on  $(0, \infty)$  (inside  $X_\Delta$ ). The  $\zeta$  is called the life time of  $\mathbf{M}$ .

For  $A \in \mathcal{F}_\infty$ ,

$$P_\mu(A) := \int_X P_x(A) \mu(dx), \quad \mu \in \mathcal{P}(S_\Delta),$$

where  $\mathcal{P}(S_\Delta)$  denotes the family of all probability measures on  $S_\Delta$ . The Markov processes  $\mathbf{M}$  is said to have the strong Markov property if for all  $\{\mathcal{F}_t\}$ -stopping times  $\sigma$ .

$$P_\mu(X_{\sigma+s} \in B | \mathcal{F}_\sigma) = P_{X_\sigma}(X_s \in B), \quad P_\mu - a.s.$$

for any  $\mu \in \mathcal{P}(S_\Delta)$ ,  $B \in \mathcal{B}$ ,  $s \geq 0$  and  $\{\mathcal{F}_t\}$ -stopping time  $\sigma$ . We say that  $\mathbf{M}$  is quasi-left continuous if for any  $\{\mathcal{F}_t\}$ -stopping time  $\sigma_n$  increasing to  $\sigma$

$$P_\mu \left( \lim_{n \rightarrow \infty} X_{\sigma_n} = X_\sigma, \sigma < \infty \right) = P_\mu(\sigma < \infty), \mu \in \mathcal{P}(S_\Delta).$$

A normal Markov process  $\mathbf{M}$  on  $(S, \mathcal{B})$  satisfying the condition **(H)** is called a *Hunt process* if there exists an admissible filtration  $\{\mathcal{F}_t\}$  such that  $\mathbf{M}$  is strong Markov and quasi-left continuous with respect to  $\{\mathcal{F}_t\}$ . In the sequel, when we state about Markov processes, they stand for Hunt processes.

Let us assume about the based Markov process  $\mathbf{M}$ .

**Assumption 2.1.**

- (I)  $\mathbf{M}$  is irreducible, that is, for any  $P_t$ -invariant set  $B$  it satisfies either  $m(B)$  or  $m(S \setminus B) = 0$ .
- (II) There exists a transition density  $p(t, x, y)$  associated with  $\mathbf{M}$ , that is, the process  $\mathbf{M}$  is absolutely continuous with respect to  $m$ .
- (III)  $\mathbf{M}$  is symmetric, that is,  $p(t, x, y) = p(t, y, x)$ .
- (IV) The transition density  $p(t, x, y)$  is jointly continuous on  $[0, \infty) \times S \times S$ .
- (V)  $\mathbf{M}$  is conservative, that is,  $P_t 1 = 1$  for any  $t > 0$ .
- (VI) The semigroup  $P_t$  has the strong Feller property, that is, for any  $f \in \mathcal{B}_b$ ,  $P_t f$  is the bounded continuous function.

In this paper, we always assume that  $S = \mathbb{R}^d$  and  $m$  is the Lebesgue measure on  $\mathbb{R}^d$  unless explicitly stated otherwise. We denote by  $G_\beta(x, y)$  the  $\beta$ -resolvent kernel, that is,

$$G_\beta(x, y) = \int_0^\infty e^{-\beta t} p(t, x, y) dt.$$

If the process  $\mathbf{M}$  is transient, there exists the 0-resolvent kernel  $G_0(x, y)$  and it is called *Green function*. We denote  $G_0(x, y)$  by  $G(x, y)$ . Hence for all measurable  $f \geq 0$ ,

$$E_x \left[ \int_0^\infty f(X_s) dx \right] = \int_X G(x, y) f(y) m(dy).$$

Let  $\mathcal{H}$  be the generator of the Markov process  $\mathbf{M}$ , that is

$$P_t = e^{t\mathcal{H}}.$$

**Definition 2.2.** A real valued process  $A_t(\omega)$ ,  $t \geq 0$  is called a *positive continuous additive functional* (in abbreviation *PCAF*) of  $\mathbf{M}$  if the following conditions are satisfied:

- (A.1)  $A_t(\omega)$  is  $\mathcal{F}_t$ -measurable.
- (A.2) There exists a set  $\Lambda \in \mathcal{F}_\infty$  such that  $P_x(\Lambda) = 1$ ,  $\forall x \in X$ ,  $\theta_t \Lambda \subset \Lambda$ , and moreover for each  $\omega \in \Lambda$ ,  $A_t(\omega)$  is positive continuous,  $A_0(\omega) = 0$ .
- (A.3)  $A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s \omega)$ ,  $\forall s, t \geq 0$ .

Let  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  be the Dirichlet form generated by  $\mathbf{M}$ , that is,

$$\begin{aligned}\mathcal{E}(u, v) &= \lim_{t \downarrow 0} \frac{1}{t} (u - P_t u, v)_m = (-\mathcal{H}u, v)_m, \\ \mathcal{D}(\mathcal{E}) &= \{u \in L^2(m) : \mathcal{E}(u, u) < \infty\},\end{aligned}$$

where  $(\cdot, \cdot)_m$  denotes the  $L^2(m)$ -inner product. The Dirichlet form theory is due to Fukushima, Oshima and Takeda's book [19]. For  $\alpha \geq 0$ , we define

$$\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha \int u^2 dx.$$

Given a Dirichlet space  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  relative  $L^2(S; m)$ , we denote by  $\mathcal{D}_e(\mathcal{E})$  the family of  $m$ -measurable functions  $u$  on  $S$  such that  $|u| < \infty$   $m$ -a.e. and there exists an  $\mathcal{E}$ -Cauchy sequence  $\{u_n\}$  of functions in  $\mathcal{D}(\mathcal{E})$  such that  $\lim_{n \rightarrow \infty} u_n = u$   $m$ -a.e. We call  $\{u_n\}$  as above an approximating sequence for  $u \in \mathcal{D}_e(\mathcal{E})$ .  $\mathcal{D}_e(\mathcal{E})$  is linear space containing  $\mathcal{D}(\mathcal{E})$ .

**Theorem 2.3** ([19] Theorem 1.5.2). *Let  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  be a Dirichlet space relative to  $L^2(S; m)$ .*

(I) *For any  $u \in \mathcal{D}_e(\mathcal{E})$  and its approximating sequence  $\{u_n\}$ , the limit*

$$\mathcal{E}(u, u) = \lim_{n \rightarrow \infty} \mathcal{E}(u_n, u_n)$$

*exists and does not depend on the choice of the approximating sequence for  $u$ .*

(II)  $\mathcal{D}(\mathcal{E}) = \mathcal{D}_e(\mathcal{E}) \cap L^2(S; m)$ .

By virtue of Theorem 2.3,  $\mathcal{E}$  can be well extended to  $\mathcal{D}_e(\mathcal{E})$  as a non-negative definite symmetric bilinear form. We call  $(\mathcal{D}_e(\mathcal{E}), \mathcal{E})$  the *extended Dirichlet space* of  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ . If the process  $\mathbf{M}$  is transient, the extended Dirichlet space  $(\mathcal{D}_e(\mathcal{E}), \mathcal{E})$  becomes Hilbert space. If the process  $\mathbf{M}$  is recurrent, the constant function  $u = 1$  belongs to  $\mathcal{D}_e(\mathcal{E})$ .

A core of a symmetric Dirichlet form  $\mathcal{E}$  is by definition a subset  $\mathcal{C}$  of  $\mathcal{D}(\mathcal{E}) \cap C_0(S)$  such that  $\mathcal{C}$  is dense in  $\mathcal{D}(\mathcal{E})$  with  $\mathcal{E}_1$ -norm and dense in  $C_0(S)$  with uniform norm.  $\mathcal{E}$  is called *regular* if  $\mathcal{E}$  possesses a core. It is well-known that if  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is a regular Dirichlet form, there exists a Hunt process associated with  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ . By [19, Example 1.2.1], if  $S = \mathbb{R}^d$ , a Dirichlet form has the following representation, so-called Beurling-Deny representation;

$$\mathcal{E}(u, v) = \sum_{i,j=1}^d \int_{\mathbb{R}^d} \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} \nu_{ij}(dx) + \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} (u(x) - u(y))(v(x) - v(y)) J(dx dy), \quad (4)$$

where  $\nu_{ij}$  ( $1 \leq i, j \leq d$ ) are Radon measures on  $\mathbb{R}^d$  such that for any  $\xi \in \mathbb{R}^d$  and any compact set  $K \subset D$ ,

$$\sum_{i,j=1}^d \xi_i \xi_j \nu_{ij}(K) \geq 0, \quad \nu_{ij}(K) = \nu_{ji}(K), \quad 1 \leq i, j \leq d.$$

$\Delta$  is diagonal in  $\mathbb{R}^d \times \mathbb{R}^d$ , that is,  $\Delta = \{(x, x) : x \in \mathbb{R}^d\}$ ,  $J$  is a positive symmetric Radon measure on the product space  $\mathbb{R}^d \times \mathbb{R}^d$  off the diagonal  $\Delta$  such that for any compact set  $K$  and open set  $D_1$  with  $K \subset D_1 \subset \mathbb{R}^d$

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} |x - y|^2 J(dx dy) < \infty, \quad J(K, \mathbb{R}^d \setminus D_1) < \infty.$$

The first term in Beurling-Deny representation (4) is called *diffusion term* and the second one is called *jump term*. Indeed, the Beurling-Deny representation has one more term, so-called *killing term*. But we only concern with conservative Markov processes in this paper. Hence we omit the third term, *killing term*. In this paper, we only consider that the diffusion term is Dirichlet integral, that is,  $\frac{1}{2}\mathbf{D}$ . For the jump term, we consider various cases.

Let  $p(\xi)$  be the symbol of the generator of the symmetric Markov process. In this paper, we assume that  $p(\xi)$  does not depend on  $x \in \mathbb{R}^d$  and radially symmetric, that is  $p(\xi) = p(|\xi|)$ .

A system of probability measures  $\{\nu_t, t > 0\}$  on  $\mathbb{R}^d$  is said to be a continuous symmetric convolution semigroup if

$$\left\{ \begin{array}{l} \nu_t * \nu_s = \nu_{t+s} \quad t, s > 0 \\ \nu_t(A) = \nu_t(-A) \quad A \in \mathcal{B}(\mathbb{R}^d) \\ \lim_{t \downarrow 0} \nu_t = \delta_0 \end{array} \right.$$

where  $\nu_s * \nu_t$  denotes the convolution between  $\nu_s$  and  $\nu_t$ , that is,  $\int_{\mathbb{R}^d} \nu_t(A - y)\nu_s(dy)$  and  $\delta_0$  is the Dirac measure concentrated at the origin. Also what the Markov process  $\mathbf{M}$  is relative to the convolution semigroup  $\nu_t$  means

$$\nu_t(A - x) = \int_A p(t, x, y)dy, \quad \text{for } A \in \mathcal{B}(\mathbb{R}^d).$$

The celebrated *Lévy-Khinchin formula* under the present symmetry assumption reads as follows:

$$\left\{ \begin{array}{l} \hat{\nu}_t(\xi) \left( = \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} \nu_t(dx) \right) = e^{-tp(\xi)}, \\ p(\xi) = \frac{1}{2}(S\xi, \xi) + \int_{\mathbb{R}^d} (1 - \cos(\langle \xi, x \rangle))J(dx), \end{array} \right.$$

where  $\langle \cdot, \cdot \rangle$  denote the Euclid inner product,  $S$  is a non-negative definite  $d \times d$  symmetric matrix.  $J$  is a symmetric measure on  $\mathbb{R}^d \setminus \{0\}$  such that

$$\int_{\mathbb{R}^d \setminus \{0\}} \frac{|x|^2}{1 + |x|^2} J(dx) < \infty.$$

Given convolution semigroup, it is known that

$$\left\{ \begin{array}{l} \mathcal{E}(u, v) = \int_{\mathbb{R}^d} \hat{u}(\xi)\bar{\hat{v}}(\xi)p(\xi)d\xi \\ \mathcal{D}(\mathcal{E}) = \{u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 p(\xi)d\xi < \infty\} \end{array} \right.$$

where  $\hat{u}$  is the Fourier transform of  $u$ , that is,

$$\hat{u}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} u(x)dx.$$

Let  $\{\nu_t, t > 0\}$  be a convolution semigroup. From the expression

$$P_t f(x) = \int_{\mathbb{R}^d} f(x+y) \nu_t(dy), \quad f \in C_\infty(\mathbb{R}^d),$$

it is easy to see that  $\{P_t, t > 0\}$  has the property:  $P_t(C_\infty) \subset C_\infty$ . Let  $\mathbf{M}$  be the Markov process generated by  $P_t$ . The process  $\mathbf{M}$  possesses the *spatial homogeneity*:

$$P_x(X_{t_1} \in E_1, \dots, X_{t_k} \in E_k) = P_0(X_{t_1} + x \in E_1, \dots, X_{t_k} + x \in E_k),$$

where  $t_1 < \dots < t_k$ ,  $E_1 \in \mathcal{B}(\mathbb{R}^d)$ ,  $\dots$ ,  $E_k \in \mathcal{B}(\mathbb{R}^d)$ .

The process  $\mathbf{M}$  is called *Lévy process* if it possesses, together condition **(H)** of sample paths, the property of stationary independent increments, that is, for all  $0 \leq s < t$  the random variable  $X_t - X_s$  is independent of  $\mathcal{F}_s$  and

$$P_{X_t - X_s} = P_{X_{t-s}}.$$

**Definition 2.4.** Denote by  $\mathcal{O}$  the family of all open subset of  $S$ . For  $A \in \mathcal{O}$  we define

$$\begin{aligned} \mathcal{L}_A &= \{u \in \mathcal{D}(\mathcal{E}) : u \geq 1 \text{ } m - \text{ a.e. on } A\} \\ \text{Cap}(A) &= \begin{cases} \inf_{u \in \mathcal{L}_A} \mathcal{E}_1(u, u), & \mathcal{L}_A \neq \emptyset \\ \infty, & \mathcal{L}_A = \emptyset \end{cases} \end{aligned}$$

for any set  $A \subset S$  we let

$$\text{Cap}(A) = \inf_{B \in \mathcal{O}, A \subset B} \text{Cap}(B).$$

We call a function  $u$  on  $S$  quasi-continuous if there exists for any  $\epsilon > 0$  an open set  $G \subset S$  such that  $\text{Cap}(G) < \epsilon$  and the restriction of  $u$  to  $S \setminus G$  is continuous. It is known that each  $u \in \mathcal{D}(\mathcal{E})$  admits a quasi-continuous version (cf. [19, Theorem 2.1.3]). From now on, every function  $u \in \mathcal{D}(\mathcal{E})$  is considered to be quasi-continuous already.

Suppose that the Dirichlet form  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is transient, that is, the process which corresponds to  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is transient. Let us define the 0-order capacity  $\text{Cap}_{(0)}(A)$  by replacing  $\mathcal{D}(\mathcal{E})$  and  $\mathcal{E}_1$  in Definition 2.4 with  $\mathcal{D}_e(\mathcal{E})$  and  $\mathcal{E}$  respectively. Then by [19, Theorem 2.1.6], a function is quasi continuous with respect to  $\text{Cap}_{(0)}$  if and only if it is quasi continuous with respect to  $\text{Cap}$ . Hence any  $u \in \mathcal{D}_e(\mathcal{E})$  admits a quasi continuous modification  $\tilde{u}$  (cf. [19, Theorem 2.1.7]). Every function  $u \in \mathcal{D}_e(\mathcal{E})$  is also considered to be quasi-continuous already.

A set  $B \subset S_\Delta$  is called nearly Borel measurable if for each  $\mu \in \mathcal{P}(S_\Delta)$  there exist Borel sets  $B_1, B_2 \in \mathcal{B}_\Delta(S)$  such that  $B_1 \subset B \subset B_2$  and  $P_\mu(X_t \in B_2 \setminus B_1, \exists t \geq 0) = 0$ .

Let  $\{P_t, t > 0\}$  be the Markovian semigroup on  $L^2(S; m)$  associated with the Dirichlet form  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ .  $u \in L^2(S; m)$  is called  $\alpha$ -excessive (with respect to  $\{P_t, t > 0\}$ ) if

$$u \geq 0, \quad e^{-\alpha t} P_t u \leq u, \quad m - \text{ a.e.}, \quad \forall t > 0.$$

When  $\alpha = 0$ , the 0-excessive function is simply called excessive (or  $P_t$ -excessive) function.

**Definition 2.5.** A nearly Borel measurable function  $u$  is finely continuous if and only if  $t \mapsto u(X_t)$  is right continuous on  $[0, \infty)$  a.s. In particular, any  $\alpha$ -excessive function is finely continuous.

We introduce the notion of subordinators.

**Definition 2.6.** A stochastic process  $T = \{T_t, t \geq 0\}$  is said to be a subordinator if it is one-dimensional Lévy process taking values in  $[0, \infty)$  with  $T_0 = 0$ . Let  $\phi$  be its Laplace exponent:

$$E[\exp(-\lambda T_t)] = \exp(-t\phi(\lambda)) \quad \text{for every } t > 0 \text{ and } \lambda > 0.$$

In this paper, when we use the subordinator, we subordinate the Brownian motion only.

**Definition 2.7.** A real-valued function  $f \in C^\infty((0, \infty))$  is called a Bernstein function if

$$f \geq 0, \quad (-1)^k \frac{d^k f(x)}{dx^k} \leq 0 \quad (5)$$

holds for all  $k \in \mathbb{N}$ .

It is known that the Laplace exponent  $\phi$  can be expressed as

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - \exp(-\lambda x))\pi(dx)$$

for some  $b \geq 0$  and Lévy measure  $\pi$  on  $(0, \infty)$  with  $\int_0^\infty (1 \wedge x)\pi(dx) < \infty$ . The Laplace exponent  $\phi$  is a Bernstein function with  $\phi(0) = 0$ . Conversely, any Bernstein function  $f$  with  $f(0) = 0$  is the Laplace exponent of a subordinator.

We introduce some examples of subordinator.

In the remainder in this section,  $\mathbf{M} = (P_x, B_t)$  is a Brownian motion in  $\mathbb{R}^d$  running twice as fast as the standard Brownian motion, that is, its generator is  $\Delta$ . Suppose that  $T := \{T_t, t \geq 0\}$  is a subordinator with Laplace exponent  $\phi$ . We suppose that  $T$  is independent of  $\mathbf{M}$ . Let  $\mathbf{M}^\phi$  be the subordinate process  $\{B_{T_t}, t \geq 0\}$ .

**Example 2.8.** Let  $\phi(\lambda) = \frac{1}{2}\lambda^{\alpha/2}$ , where  $\alpha \in (0, 2]$ . Then  $\mathbf{M}^\phi$  is called a symmetric  $\alpha$ -stable process in  $\mathbb{R}^d$ . When  $\alpha = 2$ , the subordinate process becomes the standard Brownian motion.

**Example 2.9.** Let  $\phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$  ( $m > 0$ ), where  $\alpha \in (0, 2)$  and  $m > 0$ . Then  $\mathbf{M}^\phi$  is called a relativistic  $\alpha$ -stable process in  $\mathbb{R}^d$ .

**Example 2.10.** Let  $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$ ,  $\alpha \in (0, 2]$ . The process  $\mathbf{M}^\phi$  is called a geometric  $\alpha$ -stable process for  $\alpha \in (0, 2)$  and a variance gamma process for  $\alpha = 2$ .

**Remark 2.11.** Let  $T = \{T_t, t \geq 0\}$  is a subordinator with the Laplace exponent  $\phi$  and it is independent of  $\mathbf{M}$ . The generator of the subordinate process  $\mathbf{M}^\phi$  becomes  $-\phi(-\Delta)$ .

Let  $P_t$  be a positive semigroup with integral kernel  $p(t, x, y)$ . For a positive  $P_t$ -excessive function  $h(x)$  set

$$p^h(t, x, y) = \frac{1}{h(x)}p(t, x, y)h(y), \quad t > 0, \quad x, y \in \mathbb{R}^d, \quad (6)$$

and denote by  $P_t^h$  the associated semigroup,  $P_t^h f(x) = \int_{\mathbb{R}^d} p^h(t, x, y)f(y)dy$ . Then  $p^h(t, x, y)$  becomes a transition probability density because

$$P_t^h 1(x) = \frac{1}{h(x)}P_t h(x) \leq \frac{h(x)}{h(x)} = 1.$$

We call the process generated by  $P_t^h$  the Doob's  $h$ -transformed process. This process is a  $h^2m$ -symmetric Markov process. Indeed,

$$\begin{aligned} (P_t^h f, g)_{h^2m} &= \int P_t^h f(x)g(x)h^2(x)dx \\ &= \int \frac{1}{h(x)}P_t(h(x)f(x))g(x)h^2(x)dx \\ &= \int P_t(h(x)f(x))h(x)g(x)dx \\ &= \int f(x) \cdot \frac{1}{h(x)}P_t(h(x)g(x))h^2(x)dx \quad (\text{by symmetry of } P_t) \\ &= \int f(x)P_t^h(g(x))h^2(x)dx = (f, P_t^h g)_{h^2m}. \end{aligned}$$

### 3 Some classes of measures

Now we define some classes of measures. These classes play important roles as potentials. For a measure  $\mu$ , let us denote

$$\mu_R(\cdot) = \mu(\cdot \cap B(R)), \quad \mu_{R^c} = \mu(\cdot \cap B(R)^c).$$

#### 3.1 Kato class

First we define the Kato class.

**Definition 3.1.** Let  $\mu$  be a positive Radon measure on  $\mathbb{R}^d$ .

(I) We call Kato class (in abbreviation  $\mathcal{K}_d$ ) the set of measures if the measure  $\mu$  satisfies

$$\limsup_{\beta \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_\beta(x, y)\mu(dy) = 0, \quad (7)$$

where  $G_\beta(x, y)$  is the  $\beta$ -potential density.

(II) We say that a measure  $\mu$  is locally in the Kato class ( $\mu \in \mathcal{K}_{d,loc}$  in notation), if  $\mu \in \mathcal{K}_d$

for any  $R > 0$ .

(III) The class  $\mathcal{K}_d^\infty$  is defined by the set of measures belonging Kato class and satisfying

$$\lim_{R \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \int_{|y| > R} G(x, y) \mu(dy) = 0, \quad (\text{transient case}) \quad (8)$$

$$\lim_{R \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \int_{|y| > R} G_1(x, y) \mu(dy) = 0, \quad (\text{recurrent case}) \quad (9)$$

(IV) We said that the function  $V$  belongs to  $\mathcal{K}_d$  (respectively,  $\mathcal{K}_d^\infty$ ) if the measure  $V dx \in \mathcal{K}_d$  (respectively,  $\mathcal{K}_d^\infty$ ).

**Remark 3.2.** The element of class  $\mathcal{K}_d^\infty$  is called “Green tight Kato measure”.

Now we concern with the equivalence of various definitions for Kato class.

Using the Theorem 1 and Lemma 5 in [56] and Lemma 3.1 in [25], we obtain the following proposition.

**Proposition 3.3.** If the Green function  $G(x, y)$  depends only on  $|x - y|$ , i.e.  $G(x, y) = G(|x - y|)$  and there exists a constant  $b > 1$  such that

$$\limsup_{r \rightarrow 0} \frac{G(br)}{G(r)} < 1,$$

the following assertions are equivalent.

(I)  $\mu \in \mathcal{K}_d$ .

(II)  $\limsup_{a \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq a} G(x, y) \mu(dy) = 0$ .

(III)  $\limsup_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} E_x[A_t^\mu] = 0$ .

(IV)  $\limsup_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} E_x[A_{\tau_{B(x,r)}}^\mu] = 0$ .

**Remark 3.4.** The (II) in Proposition 3.3 is alternative characterization of  $\mathcal{K}_d$  ([23]). In this remark, we only consider that the process  $\mathbf{M}$  is the Brownian motion. The Kato class  $\mathcal{K}_d$  is classically defined by the following:

$$\begin{aligned} V \in \mathcal{K}_d &\iff \sup_x \int_{|x-y| < 1} |V(y)| dy < \infty, \text{ when } d = 1 \\ &\limsup_{a \rightarrow 0} \sup_x \int_{|x-y| \leq a} \log \left( \frac{1}{|x-y|} \right) |V(y)| dy = 0, \text{ when } d = 2 \\ &\limsup_{a \rightarrow 0} \sup_x \int_{|x-y| \leq a} \frac{1}{|x-y|^{d-2}} |V(y)| dy = 0, \text{ when } d \geq 3. \end{aligned}$$

Now we provide known facts on measures in the Kato class. Let  $G_\beta \mu$  be the  $\beta$ -potential of  $\mu$ , that is,

$$G_\beta \mu(x) = \int G_\beta(x, y) \mu(dy).$$

The following theorem is a Poincaré type inequality which is derived by Stollman-Voigt [43].



**Theorem 3.5** ([43]). *Let  $\mu \in \mathcal{K}_d$ . Then for  $\beta \geq 0$ ,*

$$\int_{\mathbb{R}^d} u^2(x) \mu(dx) \leq \|G_\beta \mu\|_\infty \mathcal{E}_\beta(u, u), \quad \text{for any } u \in \mathcal{D}(\mathcal{E}). \quad (10)$$

*If  $\beta = 0$ , we can replace  $\mathcal{D}(\mathcal{E})$  by  $\mathcal{D}_e(\mathcal{E})$ .*

By the definition of Kato class, we know that

$$\lim_{\beta \rightarrow \infty} \|G_\beta \mu\|_\infty = 0. \quad (11)$$

Therefore the equation (10) shows that if the measure  $\mu$  belongs to  $\mathcal{K}_d$ , for any  $\epsilon > 0$  there exists  $M(\epsilon)$  such that

$$\int_{\mathbb{R}^d} u^2(x) \mu(dx) \leq \epsilon \mathcal{E}(u, u) + M(\epsilon) \int_{\mathbb{R}^d} u^2 dx, \quad u \in \mathcal{D}(\mathcal{E}). \quad (12)$$

### 3.2 Class $\mathcal{S}_\infty$

Next we introduce a class of some small measures which have a relation to Kato class. This class plays important role in this paper. In the sequel,  $\mathbf{M}$  is transient. In this class, the gauge and conditional gauge theorems (cf. [48], [53], [9]) holds. It is known that the gaugeability, conditional gaugeability and subcriticality of the Schrödinger type operator are equivalent.

**Definition 3.6** ([9] Definition 3.1). *Let  $\mathbf{M}$  be transient. A positive smooth measure is said to be in the class  $\mathcal{S}_\infty$  if for any  $\epsilon > 0$  there is a Borel subset  $K = K(\epsilon)$  of finite  $\mu$ -measure and a constant  $\delta = \delta(\epsilon) > 0$  such that*

$$\sup_{(x,z) \in (\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta} \int_{K^c} \frac{G(x,y)G(y,z)}{G(x,z)} \mu(dy) \leq \epsilon \quad (13)$$

*and for all measurable sets  $B \subset K$  with  $\mu(B) < \delta$ ,*

$$\sup_{(x,z) \in (\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta} \int_B \frac{G(x,y)G(y,z)}{G(x,z)} \mu(dy) \leq \epsilon. \quad (14)$$

It is known that  $\mathcal{S}_\infty \subset \mathcal{K}_d^\infty$  (cf. [11]).

We give some examples of  $\mathcal{S}_\infty$ .

**Example 3.7.** *Let  $\mathbf{M}$  be the symmetric  $\alpha$ -stable process ( $0 < \alpha \leq 2$ ) and  $d = 1$ . Then the Dirac measure at the origin  $\delta_0$  belongs to  $\mathcal{S}_\infty$ .*

**Example 3.8.** *Let  $\mathbf{M}$  be the symmetric  $\alpha$ -stable process ( $0 < \alpha \leq 2$ ) and  $d \geq 2$ . Let  $\sigma_r$  be the surface measure of a sphere with radius  $r$ ,  $S_r = \{x \in \mathbb{R}^d : |x| = r\}$ . Then  $\sigma_r \in \mathcal{S}_\infty$ .*

### 3.3 Schrödinger form, relation between PCAF and Kato class

In the sequel, we assume the following;

**Assumption 3.9.**

- (I) For  $\mu = \mu^+ - \mu^- \in \mathcal{S}_\infty - \mathcal{S}_\infty$ , the embedding of  $\mathcal{D}_e(\mathcal{E})$  to  $L^2(\mu^\pm)$  is compact.
- (II) For any compact set  $K \subset \mathbb{R}^d$ , the measure  $I_K m \in \mathcal{S}_\infty$ .

**Remark 3.10.** *The Brownian motion and symmetric  $\alpha$ -stable process are satisfied (I) of Assumption 3.9. Above two processes and the relativistic  $\alpha$ -stable process are satisfied (II) of Assumption 3.9.*

For  $\mu = \mu^+ - \mu^- \in \mathcal{K}_d - \mathcal{K}_d$ , define a symmetric bilinear form  $\mathcal{E}^\mu$  by

$$\mathcal{E}^\mu(u, u) = \mathcal{E}(u, u) + \int_{\mathbb{R}^d} u^2 d\mu, \quad u \in \mathcal{D}_e(\mathcal{E}), \quad (15)$$

Since  $\mu \in \mathcal{K}_d$  charges no set of zero capacity by [2, Theorem 3.3], the form  $\mathcal{E}^\mu$  is well defined. We see from [2, Theorem 4.1] that  $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}))$  becomes a lower semi-bounded closed symmetric form. We call  $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}))$  a *Schrödinger form*. Denote by  $\mathcal{H}^\mu$  the self-adjoint operator generated by  $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}))$ :

$$\mathcal{E}^\mu(u, v) = (\mathcal{H}^\mu u, v).$$

Let  $P_t^\mu$  be the  $L^2$ -semigroup generated by  $\mathcal{H}^\mu$ :  $P_t^\mu = \exp(-t\mathcal{H}^\mu)$ . We see from [2, Theorem 6.3(iv)] that  $P_t^\mu$  admits a symmetric integral kernel  $p^\mu(t, x, y)$  which is jointly continuous function on  $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ .

For  $\mu \in \mathcal{K}_d$ , let  $A_t^\mu$  be a positive continuous additive functional which is in the “Revuz correspondence” with  $\mu$ : for any  $f \in \mathcal{B}^+$  and  $\gamma$ -excessive function  $h$ ,

$$\langle h\mu, f \rangle = \lim_{t \rightarrow 0} \frac{1}{t} E_{hm} \left( \int_0^t f(X_s) dA_s^\mu \right), \quad (16)$$

([19, p.188]). For  $\mu = \mu^+ - \mu^- \in \mathcal{K}_d - \mathcal{K}_d$ , set  $A_t^\mu = A_t^{\mu^+} - A_t^{\mu^-}$ .

**Example 3.11.** *Let  $V$  be the function belonging the Kato class  $\mathcal{K}_d$  and  $\mu(dx) = V(x)dx$ . Then the additive functional  $A_t^\mu$  corresponding with  $\mu$  is expressed by*

$$A_t^\mu = \int_0^t V(X_s) ds.$$

**Example 3.12.** *Let  $\mathbf{M}$  be the Brownian motion, the symmetric  $\alpha$ -stable process ( $1 < \alpha \leq 2$ ) or the relativistic  $\alpha$ -stable process ( $1 < \alpha < 2$ ). Let  $d = 1$  and  $\mu = \delta_0$ , where  $\delta_0$  is the Dirac measure at the origin. Then the corresponding additive functional is the local time at the origin  $l_0(t)$ .*

By the Feynman-Kac formula, the semigroup  $P_t^\mu$  is written as

$$P_t^\mu f(x) = E_x[\exp(-A_t^\mu) f(X_t)]. \quad (17)$$

Now we introduce the time-changed process by additive functional  $A_t^\mu$ . Let  $\tau_t$  be the right continuous inverse of  $A_t^\mu$ , that is,

$$\tau_t = \inf\{s : A_s^\mu > t\}$$

with the convention that  $\inf \emptyset = \infty$ . Let  $\tilde{S} := \{x \in \mathbb{R}^d : P_x(\tau_0 = 0) = 1\}$  be the fine support of  $\mu$  and let  $S$  be the topological support of  $\mu$ . The time-changed process  $Y_t^\mu$  of  $X_t$  by  $A_t^\mu$  is defined by  $Y_t^\mu = X_{\tau_t}$ , whose state space is  $\tilde{S}$ . However, since  $\tilde{S} \subset S$  modulo a set having zero capacity, the semigroup of  $Y_t^\mu$  is  $\mu$ -symmetric and determines a strongly continuous semigroup on  $L(S, \mu)$ . The principal eigenvalue of the time-change process of this type plays important role to construct the harmonic function in §5.

## 4 Spectral functions

In this section, we define the spectral function that is a main objective in this paper. After defining the spectral function, we state properties of spectral functions.

The *spectral function*  $C(\lambda)$  is defined by the bottom of the spectrum of  $\mathcal{H}^{\lambda\mu}$ : for  $\mu = \mu^+ - \mu^- \in \mathcal{S}_\infty - \mathcal{S}_\infty$ ,

$$C(\lambda) = - \inf \left\{ \mathcal{E}^{\lambda\mu}(u, u) ; u \in \mathcal{D}(\mathcal{E}), \int_{\mathbb{R}^d} u^2 dx = 1 \right\}. \quad (18)$$

The following lemma is relation between the spectral function and the bottom of spectral of times changed process.

**Lemma 4.1.** *The following statements are equivalent.*

- (i)  $\inf \left\{ \mathcal{E}(u, u) + \int_{\mathbb{R}^d} u^2 d\mu^+ : \int_{\mathbb{R}^d} u^2 d\mu^- = 1 \right\} < 1$ .
- (ii)  $\inf \left\{ \mathcal{E}(u, u) + \int_{\mathbb{R}^d} u^2 d\mu : \int_{\mathbb{R}^d} u^2 dx = 1 \right\} < 0$ .

*Proof.* Assume (i). Then there exists a  $\varphi_0 \in C_0^\infty(\mathbb{R}^d)$  with  $\int_{\mathbb{R}^d} \varphi_0^2 d\mu^- = 1$  such that

$$\mathcal{E}(\varphi_0, \varphi_0) + \int_{\mathbb{R}^d} \varphi_0^2 d\mu^+ < 1.$$

Hence we see that

$$\mathcal{E}(\varphi_0, \varphi_0) + \int_{\mathbb{R}^d} \varphi_0^2 d\mu^+ < \int_{\mathbb{R}^d} \varphi_0^2 d\mu^-.$$

Letting

$$u_0 = \frac{\varphi_0}{\sqrt{\int_{\mathbb{R}^d} \varphi_0^2 dx}},$$

we have

$$\mathcal{E}(u_0, u_0) + \int_{\mathbb{R}^d} u_0^2 d\mu < 0.$$

Assume (ii). Then there exists a  $\psi_0 \in C_0^\infty(\mathbb{R}^d)$  with  $\int_{\mathbb{R}^d} \psi_0^2 dx = 1$  such that

$$\mathcal{E}(\psi_0, \psi_0) + \int_{\mathbb{R}^d} \psi_0^2 d\mu < 0.$$

Letting

$$u_0 = \frac{\psi_0}{\sqrt{\int_{\mathbb{R}^d} \psi_0^2 d\mu^-}},$$

we have

$$\mathcal{E}(u_0, u_0) + \int_{\mathbb{R}^d} u_0^2 d\mu^+ < 1.$$

□

**Remark 4.2.** We see from [48, Lemma 3.5] that if

$$\inf \left\{ \mathcal{E}(u, u) + \int_{\mathbb{R}^d} u^2 d\mu : \int_{\mathbb{R}^d} u^2 dx = 1 \right\} > 0,$$

then

$$\inf \left\{ \mathcal{E}(u, u) + \int_{\mathbb{R}^d} u^2 d\mu^+ : \int_{\mathbb{R}^d} u^2 d\mu^- = 1 \right\} > 1.$$

However, the converse does not hold in general. In fact, suppose that the process  $\mathbf{M}$  is the Brownian motion and  $\mu = -\sigma_R$ , the surface measure of the sphere  $\partial B(R)$ . Then the first infimum is equal to zero, while the second one is greater than 1 for  $R < \frac{d-2}{2}$  ([48]).

Define

$$\begin{aligned} \lambda^+ &= \inf\{\lambda > 0 : C(\lambda) > 0\} \\ \lambda^- &= \sup\{\lambda < 0 : C(\lambda) > 0\}. \end{aligned}$$

It follows from  $\mu^+, \mu^- \in \mathcal{S}_\infty$  that  $\lambda^+ > 0$  and  $\lambda^- < 0$  (cf. [49, Lemma 4.2]).

**Lemma 4.3.** *Let  $\mu = \mu^+ - \mu^- \in \mathcal{S}_\infty - \mathcal{S}_\infty$  with  $\mu^- \not\equiv 0$  (resp.  $\mu^+ \not\equiv 0$ ). Then the number  $\lambda^+$  (resp.  $\lambda^-$ ) is characterized as a unique positive (resp. negative) number such that*

$$\begin{aligned} \inf \left\{ \mathcal{E}(u, u) + \lambda^+ \int_{\mathbb{R}^d} u^2 d\mu^+ : \lambda^+ \int_{\mathbb{R}^d} u^2 d\mu^- = 1 \right\} &= 1. \\ \left( \text{resp. } \inf \left\{ \mathcal{E}(u, u) - \lambda^- \int_{\mathbb{R}^d} u^2 d\mu^- : -\lambda^- \int_{\mathbb{R}^d} u^2 d\mu^+ = 1 \right\} \right) &= 1. \end{aligned} \tag{19}$$

**Remark 4.4.** Since  $\mathcal{H}^{\lambda-\mu} = \mathcal{H} + (-\lambda^-)\mu^- - (-\lambda^-)\mu^+$ , we only consider  $\lambda^+$  from now on.

*proof of lemma 4.3.* Let  $\mathbb{R}^d = F + F^c$  be the Hahn decomposition:  $\mu^-(F) = \mu^-(\mathbb{R}^d)$ ,  $\mu^+(F^c) = \mu^+(\mathbb{R}^d)$ . Take  $R > 0$  so large that  $\mu^-(F \cap B(R)) > 0$  and let  $A = F \cap B(R)$ . Take a sequence of non-negative functions  $f_n$  in  $C_0^\infty(\mathbb{R}^d)$  such that

$$\int_{\mathbb{R}^d} (I_A(x) - f_n(x))^2 |\mu|(dx) \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

It then holds that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n^2(x) \mu^-(dx) = \mu^-(A) > 0, \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n^2(x) \mu^+(dx) = \mu^+(A) = 0,$$

and consequently, there exists a function  $f \in C_0^\infty(\mathbb{R}^d)$  such that

$$\int_{\mathbb{R}^d} f^2(x) \mu^-(dx) = 1, \quad \int_{\mathbb{R}^d} f^2(x) \mu^+(dx) < 1. \quad (20)$$

Set

$$F(\lambda) = \inf \left\{ \mathcal{E}(u, u) + \lambda \int_{\mathbb{R}^d} u^2(x) \mu^+(dx) : \int_{\mathbb{R}^d} u^2(x) \mu^-(dx) = 1 \right\}. \quad (21)$$

First we find that  $F(0) > 0$  ( $F(0)$  is the bottom of the spectrum of the time changed process of  $\mathbf{M}$  by the additive functional  $A_t^{\mu^-}$  (cf. [46, Lemma 3.1])). Indeed, since the embedding of  $\mathcal{D}_e(\mathcal{E})$  to  $L^2(\mu^-)$  is compact, there exists the function  $u_0$  in  $\mathcal{D}_e(\mathcal{E})$  that attains the infimum of (21). If  $F(0) = 0$ , then  $\mathcal{E}(u_0, u_0) = 0$ , and thus  $u_0 = 0$  because  $(\mathcal{D}_e(\mathcal{E}), \mathcal{E})$  is a Hilbert space. This contradicts that  $\int_{\mathbb{R}^d} u_0^2 d\mu^- = 1$ . Moreover,  $F(\lambda)$ ,  $\lambda \geq 0$ , is a concave function by the definition and dominated by the function  $G(\lambda) := \mathcal{E}(f, f) + \lambda \int_{\mathbb{R}^d} f^2(x) \mu^+(dx)$ , where  $f$  is a function satisfying (20).

These properties of  $F$  show that there exists a unique  $\lambda^0 > 0$  such that  $F(\lambda^0) = \lambda^0$ . We see from Lemma 4.1 that  $\lambda^0 = \lambda^+$  and thus  $F(\lambda^+)/\lambda^+ = 1$ , which leads us the lemma.  $\square$

The shape of the spectral function  $C(\lambda)$  is the following:

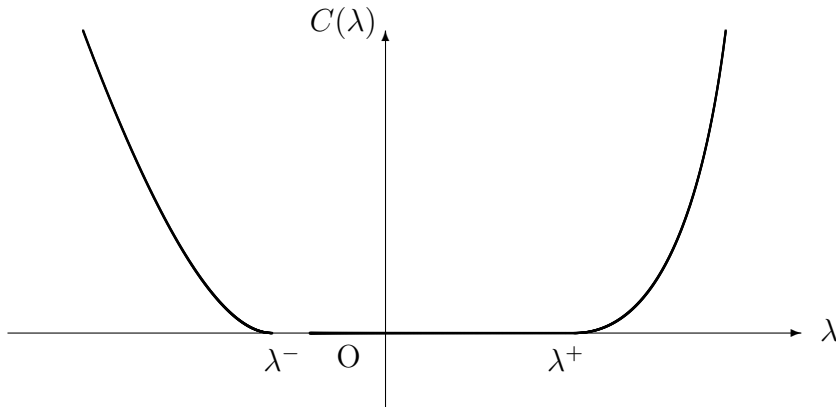


Fig. spectral function.

## 5 Ground states and criticality

In this section, we construct harmonic functions (ground state) with respect to general symmetric Markov processes including local and non-local types.

We define a  $\mathcal{H}^\mu$ -harmonic functions probabilistically.

**Definition 5.1.** *A function  $h$  on  $\mathbb{R}^d$  is said to be  $\mathcal{H}^\mu$ -harmonic in a domain  $U$ , if for any relatively compact domain  $D \subset U$ ,*

$$h(x) = E_x[\exp(-A_{\tau_D}^\mu)h(X_{\tau_D})], \quad x \in D \quad (22)$$

where  $\tau_D$  is the first exit time from  $D$ . When  $h$  is  $\mathcal{H}^\mu$ -harmonic on  $\mathbb{R}^d$ , we write  $\mathcal{H}^\mu h = 0$ .

Next, we define the criticality of Schrödinger type operators.

**Definition 5.2.** (I) *The operator  $\mathcal{H}^\mu$  is said to be subcritical, if  $\mathcal{H}^\mu$  possesses the minimal positive Green function  $G^\mu(x, y)$ , that is,*

$$G^\mu(x, y) = \int_0^\infty p^\mu(t, x, y)dt < \infty, \quad x \neq y.$$

(II) *The operator  $\mathcal{H}^\mu$  is said to be critical, if  $\mathcal{H}^\mu$  does not have the Green function but possesses the harmonic function.*

(III) *The operator  $\mathcal{H}^\mu$  is said to be supercritical if it is neither subcritical nor critical.*

Now we introduce the notion of *extended Schrödinger form*. Assume that  $\mathcal{H}^\mu$  is subcritical or critical and let  $h$  be a positive  $\mathcal{H}^\mu$ -harmonic function. We denote by  $\mathcal{D}_e(\mathcal{E}^\mu)$  the family of  $m$ -measurable function  $u$  on  $\mathbb{R}^d$  such that  $|u| < \infty$   $m$ -a.e. and there exists an  $\mathcal{E}^\mu$ -Cauchy sequence  $\{u_n\}$  of functions in  $\mathcal{D}(\mathcal{E})$  such that  $\lim_{n \rightarrow \infty} u_n = u$   $m$ -a.e. We call  $\{u_n\}$  as above an *approximating sequence* for  $u \in \mathcal{D}_e(\mathcal{E}^\mu)$ .

Note that the Dirichlet form  $(\mathcal{E}^{\mu, h}, \mathcal{D}(\mathcal{E}^{\mu, h}))$  associated with the Markov semigroup  $P_t^{\mu, h}$  is given by

$$\begin{aligned} \mathcal{E}^{\mu, h}(u, v) &= \mathcal{E}^\mu(hu, hv) \\ \mathcal{D}(\mathcal{E}^{\mu, h}) &= \{u \in L^2(\mathbb{R}^d; h^2 dx) : hu \in \mathcal{D}(\mathcal{E}^\mu)\}. \end{aligned}$$

Then we see that  $u \in \mathcal{D}_e(\mathcal{E}^\mu)$  if and only if  $\frac{u}{h} \in \mathcal{D}_e(\mathcal{E}^{\mu, h})$ , where  $\mathcal{D}_e(\mathcal{E}^{\mu, h})$  is the extended Dirichlet space of  $(\mathcal{E}^{\mu, h}, \mathcal{D}(\mathcal{E}^{\mu, h}))$ . Consequently, the Schrödinger form  $\mathcal{E}^\mu$  can be well extended to  $\mathcal{D}_e(\mathcal{E}^\mu)$  as a closed symmetric form; for  $u \in \mathcal{D}_e(\mathcal{E}^\mu)$  and its approximating sequence  $\{u_n\}$

$$\mathcal{E}^\mu(u, u) = \lim_{n \rightarrow \infty} \mathcal{E}^\mu(u_n, u_n), \quad u \in \mathcal{D}_e(\mathcal{E}^\mu) \quad (23)$$

(see [19, p.35]). We call  $(\mathcal{E}^\mu, \mathcal{D}_e(\mathcal{E}^\mu))$  the *extended Schrödinger space*. Note that in the definition of  $\mathcal{D}_e(\mathcal{E}^\mu)$ , the condition for  $\{u_n\}$  being an  $\mathcal{E}^\mu$ -Cauchy sequence can be replaced by

$$\sup_n \mathcal{E}^\mu(u_n, u_n) < \infty$$

(cf. [40, Definition 1.6]).

If  $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}))$  is a subcritical Schrödinger form, that is, the associated operator  $\mathcal{H}^\mu$  be subcritical, then  $(\mathcal{E}^\mu, \mathcal{D}_e(\mathcal{E}^\mu))$  becomes a Hilbert space by [19, Lemma 1.5.5]. In particular, a positive  $\mathcal{H}^\mu$ -harmonic function  $h$  does not belong to  $\mathcal{D}_e(\mathcal{E}^\mu)$ . If  $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}))$  is a critical Schrödinger form, that is, the associated operator  $\mathcal{H}^\mu$  be critical, its ground state  $h$  belongs to  $\mathcal{D}_e(\mathcal{E}^\mu)$  on account of [19, Theorem 1.6.3]. Noting that for  $\mu = \mu^+ - \mu^- \in \mathcal{S}_\infty - \mathcal{S}_\infty$

$$\mathcal{E}^\mu(u, u) \leq (1 + \|G|\mu|\|_\infty)\mathcal{E}(u, u)$$

by Theorem 3.5, we see that  $\mathcal{D}_e(\mathcal{E}^\mu)$  includes  $\mathcal{D}_e(\mathcal{E})$ .

There is a test for subcriticality of some Schrödinger type operators.

It was shown in [48] that the following condition is a necessary and sufficient one for a operator  $\mathcal{H}^\mu$  being subcritical:

**Theorem 5.3** ([48] Theorem 3.9). *Let  $\mu = \mu^+ - \mu^- \in \mathcal{S}_\infty - \mathcal{S}_\infty$ . Then  $\mathcal{H}^\mu$  is subcritical if and only if*

$$\inf \left\{ \mathcal{E}(u, u) + \int_{\mathbb{R}^d} u^2 d\mu^+ : \int_{\mathbb{R}^d} u^2 d\mu^- = 1 \right\} > 1.$$

**Remark 5.4.** *The left-hand-side of the above inequality is the bottom of spectrum of the time-changed process with respect to  $\mu^-$  of the subprocess  $\mathbf{M}^{\mu^+}$  by the multiplicative functional  $\exp(-A_t^{\mu^+})$ . This theorem characterize the subcriticality of Schrödinger type operator by the bottom of spectrum of the time-changed process.*

Recall that the operator  $\mathcal{H}^{\lambda^+ \mu}$  is not subcritical since

$$\inf \left\{ \mathcal{E}(u, u) + \lambda^+ \int_{\mathbb{R}^d} u^2 d\mu^+ : \lambda^+ \int_{\mathbb{R}^d} u^2 d\mu^- = 1 \right\} = 1.$$

For a non-negative bounded Borel function  $w \not\equiv 0$  with compact support, define  $\nu = \lambda^+ \mu + w dx$ . We then see from [48] that  $\mathcal{H}^\nu$  is subcritical and its Green function  $G^\nu(x, y)$  is equivalent to  $G$ : there exist positive constants  $c, C$  such that

$$cG(x, y) \leq G^\nu(x, y) \leq CG(x, y) \quad \text{for } x \neq y. \quad (24)$$

Let  $G^\nu$  be the Green operator,  $G^\nu f(x) = \int_{\mathbb{R}^d} G^\nu(x, y)f(y)dy$ .

**Lemma 5.5.** *For a positive function  $\varphi \in C_0(\mathbb{R}^d)$ ,  $G^\nu \varphi$  belongs to  $\mathcal{D}_e(\mathcal{E}^\nu)$*

*Proof.* Let  $G_\beta^\nu$  be  $\beta$ -resolvent of  $\mathcal{H}^\nu$ . Then  $G_\beta^\nu \varphi$  belongs to  $\mathcal{D}(\mathcal{E})$  and  $G_\beta^\nu \varphi \uparrow G^\nu \varphi$  as  $\beta \rightarrow 0$ . Moreover, by (24)

$$\begin{aligned} \mathcal{E}^\nu(G_\beta^\nu \varphi, G_\beta^\nu \varphi) &\leq \mathcal{E}_\beta^\nu(G_\beta^\nu \varphi, G_\beta^\nu \varphi) = (\varphi, G_\beta^\nu \varphi) \\ &\leq (\varphi, G^\nu \varphi) \leq C(\varphi, G\varphi) < \infty, \end{aligned}$$

which proves the lemma. □

Now we construct a  $\mathcal{H}^{\lambda^+\mu}$ -harmonic function. We follow our paper [52]. Since the embedding of  $\mathcal{D}_e(\mathcal{E})$  to  $L^2(\mu)$  is compact. Then there exists a function  $u_0 \in \mathcal{D}_e(\mathcal{E})$  such that  $u_0$  attains the infimum:

$$\inf \left\{ \mathcal{E}(u, u) + \lambda^+ \int_{\mathbb{R}^d} u^2 d\mu^+ : u \in \mathcal{D}_e(\mathcal{E}), \lambda^+ \int_{\mathbb{R}^d} u^2 d\mu^- = 1 \right\} = 1. \quad (25)$$

Indeed, let  $\{u_n\}$  be a sequence of  $\mathcal{D}_e(\mathcal{E})$  such that  $u_n$  converges to  $u_0 \in \mathcal{D}_e(\mathcal{E})$  in the sense of  $\mathcal{E}$ -weakly and

$$\begin{cases} \int_{\mathbb{R}^d} u_n^2 d\mu^- = 1, \text{ for any } n, \\ \lim_{n \rightarrow \infty} \left\{ \mathcal{E}(u_n, u_n) + \lambda^+ \int_{\mathbb{R}^d} u_n^2 d\mu^+ \right\} = 1. \end{cases}$$

We take a subsequence of  $\{n\}$  if necessary here. By the Banach-Steinhaus theorem and the assumption of the compact embedding from  $\mathcal{D}_e(\mathcal{E})$  to  $L^2(\mu^+)$ , we find that

$$1 = \liminf_{n \rightarrow \infty} \left\{ \mathcal{E}(u_n, u_n) + \lambda^+ \int_{\mathbb{R}^d} u_n^2 d\mu^+ \right\} \geq \mathcal{E}(u_0, u_0) + \lambda^+ \int_{\mathbb{R}^d} u_0^2 d\mu^+ \geq 1.$$

Moreover since the embedding from  $\mathcal{D}_e(\mathcal{E})$  to  $L^2(\mu^-)$  is compact, we have

$$1 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} u_n^2 d\mu^- = \int_{\mathbb{R}^d} u_0^2 d\mu^-.$$

Therefore we know that the function  $u_0$  attains the infimum (25).

Let  $\mathbf{M}^{\lambda^+\mu^+}$  be the subprocess of  $\mathbf{M}$  by the multiplicative functional  $\exp(-\lambda^+ A_t^{\mu^+})$ . Then the function  $u_0$  is the first eigenfunction corresponding to the generator of the time changed process of  $\mathbf{M}^{\lambda^+\mu^+}$  by  $A_t^{\lambda^+\mu^-}$ . The time changed process is irreducible because  $\int_{\mathbb{R}^d} G^{\lambda^+\mu^+}(x, y) \mu^-(dy) > 0$ . Hence  $u_0 > 0$   $\mu^-$ -a.e by [15, Theorem 7.3].

**Lemma 5.6.** *Let  $u_0$  be the function in (25). Then the measure  $u_0 \mu^-$  is of finite energy integral with respect to  $\mathcal{E}^{\lambda^+\mu^+}$ .*

*Proof.* Let  $f \in \mathcal{D}_e(\mathcal{E})$ . Then

$$\int_{\mathbb{R}^d} f(x) u_0(x) \mu^-(dx) \leq \left( \int_{\mathbb{R}^d} u_0^2(x) \mu^-(dx) \right)^{1/2} \left( \int_{\mathbb{R}^d} f^2(x) \mu^-(dx) \right)^{1/2},$$

and the right hand side is dominated by

$$C \mathcal{E}(f, f)^{1/2} \leq C \mathcal{E}^{\lambda^+\mu^+}(f, f)^{1/2}$$

by Theorem 3.5. □

The function  $u_0$  is also characterized by the equation:

$$\mathcal{E}(u_0, f) + \lambda^+ \int_{\mathbb{R}^d} u_0 f d\mu^+ = \lambda^+ \int_{\mathbb{R}^d} u_0 f d\mu^-, \quad \text{for all } f \in \mathcal{D}_e(\mathcal{E}). \quad (26)$$



Hence we see from Lemma 5.6 that

$$\mathcal{E}^{\lambda^+\mu^+}(u_0, f) = \lambda^+ \int_{\mathbb{R}^d} u_0 f d\mu^- = \mathcal{E}^{\lambda^+\mu^+}(\lambda^+ G^{\lambda^+\mu^+}(u_0\mu^-), f),$$

and

$$\begin{aligned} u_0(x) &= \lambda^+ \int_{\mathbb{R}^d} G^{\lambda^+\mu^+}(x, y) u_0(y) d\mu^-(dy) \\ &= E_x \left[ \int_0^\infty \exp(-\lambda^+ A_t^{\mu^+}) u_0(X_t) dA_t^{\lambda^+\mu^-} \right], \quad m\text{-a.e.} \end{aligned}$$

Now we set

$$h(x) = E_x \left[ \int_0^\infty \exp(-\lambda^+ A_t^{\mu^+}) u_0(X_t) dA_t^{\lambda^+\mu^-} \right]. \quad (27)$$

and prove that the function  $h$  is a bound continuous  $\mathcal{H}^{\lambda^+\mu}$ -harmonic function. We remark that  $h$  is equal to  $u_0$  q.e. and is strictly positive because  $G^{\lambda^+\mu^+}(x, y) > 0$ .

**Lemma 5.7.** *The function  $h$  is finely continuous.*

*Proof.* By the Markov property,

$$\begin{aligned} h(X_s) &= E_{X_s} \left[ \int_0^\infty \exp(-\lambda^+ A_t^{\mu^+}) u_0(X_t) dA_t^{\lambda^+\mu^-} \right] \\ &= E_x \left[ \int_0^\infty \exp(-\lambda^+ A_t^{\mu^+}(\theta_s)) u_0(X_{t+s}) dA_t^{\lambda^+\mu^-}(\theta_s) \middle| \mathcal{F}_s \right] \\ &= \exp(\lambda^+ A_s^{\mu^+}) E_x \left[ \int_0^\infty \exp(-\lambda^+ A_t^{\mu^+}) u_0(X_t) dA_t^{\lambda^+\mu^-} \middle| \mathcal{F}_s \right] \\ &\quad - \exp(\lambda^+ A_s^{\mu^+}) \int_0^s \exp(-\lambda^+ A_t^{\mu^+}) u_0(X_t) dA_t^{\lambda^+\mu^-}. \end{aligned}$$

Since the first term of right hand side is right continuous because of the right continuity of  $\mathcal{F}_s$ , we see that  $h$  is finely continuous by [19, Theorem A.2.7].  $\square$

Note that if  $h(x) = u_0(x)$   $m$ -a.e., then  $h(x) = u_0(x)$  q.e. by [19, Lemma 4.1.5]. Hence [19, Theorem 4.1.2] proves the next lemma.

**Lemma 5.8.** *The function  $h$  is strictly positive and satisfies*

$$h(x) = E_x \left[ \int_0^\infty \exp(-\lambda^+ A_t^{\mu^+}) h(X_t) dA_t^{\lambda^+\mu^-} \right] \quad (28)$$

for all  $x \in \mathbb{R}^d$ .

**Lemma 5.9.** *The function  $h$  is  $P_t^{\lambda^+\mu}$ -excessive.*

*Proof.* Set  $M_t = E_x[\int_0^\infty \exp(-\lambda^+ A_s^{\mu^+}) h(X_s) dA_s^{\lambda^+ \mu^-} | \mathcal{F}_t]$ . Then we had in the proof of Lemma 5.7

$$\exp(-\lambda^+ A_t^{\mu^+}) h(X_t) = M_t + \int_0^t \exp(-\lambda^+ A_s^{\mu^+}) u_0(X_s) dA_s^{\lambda^+ \mu^-}.$$

Hence by Ito's formula we obtain

$$\begin{aligned} \exp(-\lambda^+ A_t^\mu) h(X_t) &= \exp(\lambda^+ A_t^{\mu^-}) (\exp(-\lambda^+ A_t^{\mu^+}) h(X_t)) \\ &= h(X_0) + \int_0^t \exp(\lambda^+ A_s^{\mu^-}) dM_s - \int_0^t \exp(-\lambda^+ A_s^\mu) h(X_s) dA_s^{\lambda^+ \mu^-} \\ &\quad + \int_0^t \exp(-\lambda^+ A_s^{\mu^+}) h(X_s) \exp(\lambda^+ A_s^{\mu^-}) dA_s^{\lambda^+ \mu^-} \\ &= h(X_0) + \int_0^t \exp(\lambda^+ A_s^{\mu^-}) dM_s, \end{aligned}$$

Taking the expectation, we know that

$$E_x[\exp(-\lambda^+ A_t^\mu) h(X_t)] = E_x[h(X_0)] + E_x \left[ \int_0^t \exp(\lambda^+ A_s^{\mu^-}) dM_s \right].$$

Since the integrand in the second term is a martingale, we show that

$$E_x[\exp(-\lambda^+ A_t^\mu) h(X_t)] \leq h(x).$$

□

We see from Lemma 5.9 that the  $h$ -transformed semigroup  $P_t^{\lambda^+ \mu, h}$  generates a  $h^2 m$ -symmetric Markov process with the state space  $S^h = \{x \in \mathbb{R}^d : h(x) < \infty\}$ . Let us denote by  $\mathbf{M}^{\lambda^+ \mu, h}$  the Markov process generated by  $P_t^{\lambda^+ \mu, h}$ . Then because of non-subcriticality of  $\mathcal{H}^{\lambda^+ \mu}$ ,  $\mathbf{M}^{\lambda^+ \mu, h}$  is recurrent, in particular, conservative,  $P_t^{\lambda^+ \mu, h} \mathbf{1} = 1$ , as a result, the function  $h$  is  $P_t^{\lambda^+ \mu}$ -invariant:

$$P_t^{\lambda^+ \mu} h = h. \tag{29}$$

**Lemma 5.10.** *Finely continuous  $P_t^{\lambda^+ \mu}$ -excessive function is unique up to constant multiplication.*

*Proof.* We follow the argument in [31, P.149, Theorem 3.4]. Let  $h, h'$  be finely continuous  $P_t^{\lambda^+ \mu}$ -excessive functions. Since

$$E_x \left[ \exp(-\lambda^+ A_t^\mu) h(X_t) \left( \frac{h'}{h} \right) (X_t) \right] \leq h \cdot \frac{h'}{h}(x),$$

we have

$$E_x^{\lambda^+ \mu, h} \left[ \frac{h'}{h}(X_t) \right] \leq \frac{h'}{h}(x).$$

Now for  $y \in \mathbb{R}^d$  and  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , we put  $U_{\epsilon_n}(y) = \{z : |h(z) - h(y)| < \epsilon_n\}$ . Since  $U_{\epsilon_n}(y)$  is finely open,  $\sigma_{U_{\epsilon_n}(y)} < \infty$ ,  $P_x^{\lambda^+\mu, h}$ -a.s [19, Problem 4.6.3]. Denote  $\sigma_n = \sigma_{U_{\epsilon_n}(y)}$ . Replacing  $t$  by  $\sigma_n$ , we have

$$E_x^{\lambda^+\mu, h} \left[ \frac{h'}{h}(X_{\sigma_n}) \right] \leq \frac{h'}{h}(x). \quad (30)$$

Noting that the left hand side of (30) converges to  $\frac{h'}{h}(y)$  as  $n \rightarrow \infty$ , we obtain by Fatou's lemma

$$\begin{aligned} \frac{h'}{h}(y) &= E_x^{\lambda^+\mu, h} \left[ \liminf_{n \rightarrow \infty} \frac{h'}{h}(X_{\sigma_n}) \right] \\ &\leq \liminf_{n \rightarrow \infty} E_x^{\lambda^+\mu, h} \left[ \frac{h'}{h}(X_{\sigma_n}) \right] \\ &\leq \frac{h'}{h}(x). \end{aligned}$$

Since  $x$  and  $y$  are arbitrary,  $h'/h$  must be a constant function.  $\square$

The next theorem is first obtained by Murata [28, Theorem 2.2] when the process  $\mathbf{M}$  is Brownian motion. Using a probabilistic argument, we extend the theorem to symmetric Markov process satisfying our assumptions.

**Theorem 5.11.** *For  $w \in C_0(\mathbb{R}^d)$  with  $w \geq 0$ ,  $w \not\equiv 0$ , let  $\nu = \lambda^+\mu + wdx$ . The function  $h$  defined in (27) satisfies*

$$h(x) = \int_{\mathbb{R}^d} G^\nu(x, y) h(y) w(y) dy. \quad (31)$$

*Proof.* Note  $h$  satisfies (25) and thus

$$\mathcal{E}^\nu(h, f) = \int_{\mathbb{R}^d} h f w dx, \quad \text{for any } f \in \mathcal{D}_e(\mathcal{E}^\nu).$$

by the definition of the extended Schrödinger space. Since  $G^\nu \varphi \in \mathcal{D}_e(\mathcal{E}^\nu)$  for any  $\varphi \in C_0(\mathbb{R}^d)$  by Lemma 5.5, we obtain, by substituting  $G^\nu \varphi$  for  $f$

$$\int_{\mathbb{R}^d} h(x) \varphi(x) dx = \int_{\mathbb{R}^d} h(x) w(x) G^\nu \varphi(x) dx = \int_{\mathbb{R}^d} G^\nu(hw)(x) \varphi(x) dx$$

and thus

$$h(x) = \int_{\mathbb{R}^d} G^\nu(x, y) h(y) w(y) dy, \quad m\text{-a.e.}$$

By the same argument as in lemma 5.7, in the equation above "m-a.e.  $x$ " can be replaced by "any  $x$ ".  $\square$

**Lemma 5.12.** *The function  $h$  is bounded.*

*Proof.* Since  $h$  is finely continuous, we can find a compact set  $K$  such that  $h \leq c$  on  $K$ . Let  $\nu = \mu + I_K(x)dx$ . Note that  $\nu$  belongs to  $\mathcal{S}_\infty$ . Theorem 5.11 says that  $h$  satisfies

$$h(x) = \int_{\mathbb{R}^d} G^\nu(x, y)h(y)I_K(y)dy.$$

Since  $G^\nu(x, y)$  is equivalent to  $G(x, y)$  by [48], it holds that

$$h(x) \leq c \int_{\mathbb{R}^d} G^\nu(x, y)I_K(y)dy \leq C' \int_{\mathbb{R}^d} G(x, y)I_K(y)dy.$$

The right hand side of above inequality is bounded because  $I_K(y)dy \in \mathcal{S}_\infty$ .  $\square$

**Proposition 5.13.** *The function  $h$  is  $\mathcal{H}^{\lambda^+ \mu}$ -harmonic function; for any bounded domain  $D$*

$$E_x[\exp(-\lambda^+ A_{\tau_D}^\mu)h(X_{\tau_D})] = h(x), x \in D.$$

*Proof.* Let

$$M_t = \exp(-\lambda^+ A_t^\mu)h(X_t).$$

Then  $M_t$  is a martingale. In fact, by the Markov property and (29)

$$\begin{aligned} E_x[M_t | \mathcal{F}_s] &= \exp(-\lambda^+ A_s^\mu)E_{X_s}[\exp(-\lambda^+ A_{t-s}^\mu)h(X_{t-s})] \\ &= \exp(-\lambda^+ A_t^\mu)h(X_s) \end{aligned}$$

Hence we see that the right hand of (32) equals to  $\exp(-\lambda^+ A_s^\mu)h(X_s)$ . On account of the optional stopping theorem,

$$E_x[\exp(-\lambda^+ A_{t \wedge \tau_D}^\mu)h(X_{t \wedge \tau_D})] = h(x), \quad (32)$$

where  $D$  is a bounded domain of  $\mathbb{R}^d$ .

On the other hand, by the definition of  $\lambda^+$

$$\inf \left\{ \mathcal{E}^{\lambda^+ \mu}(u, u) : u \in C_0^\infty(D), \int_D u^2 dx = 1 \right\} > 0.$$

Hence  $\lambda^+ \mu$  is gaugeable on  $D$ , that is

$$\sup_{x \in D} E_x [\exp(-\lambda^+ A_{\tau_D})] < \infty$$

(cf. [9],[53]). We then see from [9, Corollary 2.9] that

$$\sup_{x \in D} E_x \left[ \sup_{0 \leq t \leq \tau_D} \exp(-\lambda^+ A_t^\mu) \right] < \infty. \quad (33)$$

Noting that

$$|\exp(-\lambda^+ A_{t \wedge \tau_D}) h(X_{t \wedge \tau_D})| \leq \|h\|_\infty \left( \sup_{0 \leq t \leq \tau_D} \exp(-\lambda^+ A_t^\mu) \right), \quad (34)$$

we have

$$\lim_{t \rightarrow \infty} E_x[\exp(-\lambda^+ A_{t \wedge \tau_D}^\mu) h(X_{t \wedge \tau_D})] = E_x[\exp(-\lambda^+ A_{\tau_D}^\mu) h(X_{\tau_D})]$$

on account of the quasi-left continuity of  $\mathbf{M}$ .  $\square$

We often call the function  $h$  satisfying  $\mathcal{H}^{\lambda^+ \mu} h = 0$  “ground state”.

**Lemma 5.14.** *The function  $h$  satisfies*

$$h(x) = E_x[h(X_{\tau_D})] - \lambda^+ E_x \left[ \int_0^{\tau_D} h(X_s) dA_s^\mu \right]. \quad (35)$$

*Proof.* Since  $h$  is  $\mathcal{H}^{\lambda^+ \mu}$ -harmonic, for a bounded domain  $D$ ,

$$\lambda^+ E_x \left[ \int_0^{\tau_D} h(X_t) dA_t^\mu \right] = \lambda^+ E_x \left[ \int_0^{\tau_D} E_{X_t}(\exp(-\lambda^+ A_{\tau_D}^\mu) h(X_{\tau_D})) dA_t^\mu \right].$$

By the Markov property the right hand side equals to

$$\begin{aligned} & \lambda^+ E_x \left[ \int_0^{\tau_D} \exp(\lambda^+ A_t^\mu - \lambda^+ A_{\tau_D}^\mu) h(X_{\tau_D}) dA_t^\mu \right] \\ &= E_x \left[ \exp(-\lambda^+ A_{\tau_D}^\mu) h(X_{\tau_D}) (\exp(\lambda^+ A_{\tau_D}^\mu) - 1) \right] \\ &= E_x[h(X_{\tau_D})] - E_x \left[ \exp(-\lambda^+ A_{\tau_D}^\mu) h(X_{\tau_D}) \right], \end{aligned}$$

which implies (35).  $\square$

## 6 Differentiability of spectral functions

### 6.1 Assumptions

To prove the differentiability of spectral functions, let us organize assumptions. We denote by  $h$  the ground state of  $\mathcal{H}^{\lambda^+ \mu}$

**Assumption 6.1.**

(I) *Assumption 2.1.*

(II) *Assumption 3.9.*

(III) *The function  $h$  is continuous.*

If ground states of critical operator  $\mathcal{H}^{\lambda^\pm \mu}$  belong to  $L^2(m)$ , we call it *positive critical*. If the ground state does not belong to  $L^2(m)$ , we call it *null-critical*.

Indeed, the null-criticality plays crucial rule in the proof of the differentiability of spectral functions. Moreover, we can prove that the spectral function is not differentiable if  $\mathcal{H}^{\lambda^\pm \mu}$  is positive critical.

## 6.2 An extension of Oshima's inequality

In this section, we prove a functional inequality for critical Schrödinger forms. This inequality is regarded as a version of Oshima's inequality and plays a crucial role for the proof of the differentiability of  $C(\lambda)$ .

**Lemma 6.2.** *Let  $h$  be the  $\mathcal{H}^{\lambda+\mu}$ -harmonic function constructed in the previous section. Then the  $h$ -transformed semigroup  $P_t^{\lambda+\mu, h}$  of  $P_t^{\lambda+\mu}$  has the strong Feller property.*

*Proof.* We follow the argument in [16, Corollary 5.2.7]. Let  $f$  be a bounded Borel function and  $\{x_n\}$  a sequence so that  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . Recall that  $p^{\lambda+\mu}(t, x, y)$  is jointly continuous ([2, Theorem 3.10]). Then by Fatou's lemma and the continuity of  $h$ ,

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} \frac{1}{h(x_n)} p^{\lambda+\mu}(t, x_n, y) h(y) (\|f\|_\infty \pm f(y)) dy \\ & \geq \int_{\mathbb{R}^d} \frac{1}{h(x)} p^{\lambda+\mu}(t, x, y) h(y) (\|f\|_\infty \pm f(y)) dy, \end{aligned}$$

and thus the function,

$$x \mapsto \int_{\mathbb{R}^d} \frac{1}{h(x)} p^{\lambda+\mu}(t, x, y) h(y) (\|f\|_\infty \pm f(y)) dy,$$

is lower semi-continuous. Note that  $P_t^{\lambda+\mu, h}$  is recurrent, in particular, conservative. Then

$$\begin{aligned} & \int_{\mathbb{R}^d} \frac{1}{h(x)} p^{\lambda+\mu}(t, x, y) h(y) f(y) dy \\ & = \int_{\mathbb{R}^d} \frac{1}{h(x)} p^{\lambda+\mu}(t, x, y) h(y) (\|f\|_\infty + f(y)) dy - \|f\|_\infty \\ & = - \int_{\mathbb{R}^d} \frac{1}{h(x)} p^{\lambda+\mu}(t, x, y) h(y) (\|f\|_\infty - f(y)) dy + \|f\|_\infty, \end{aligned}$$

and thus the function

$$x \mapsto \int_{\mathbb{R}^d} \frac{1}{h(x)} p^{\lambda+\mu}(t, x, y) h(y) f(y) dy,$$

is lower and upper semi-continuous. □

**Proposition 6.3.** *The  $h$ -transformed process  $\mathbf{M}^{\lambda+\mu, h} = (P_x^{\lambda+\mu, h}, X_t)$  is Harris recurrent, that is, for a non-negative function  $f$ ,*

$$\int_0^\infty f(X_t) dt = \infty \quad P_x^{\lambda+\mu, h} \text{-a.s.} \quad (36)$$

whenever  $m(\{x : f(x) > 0\}) > 0$ .

*Proof.* Since  $P_t^{\lambda^+\mu,h}$  generates an  $h^2m$ -symmetric recurrent Markov process,

$$P_x[\sigma_A \circ \theta_n < \infty, \forall n \geq 0] = 1 \text{ for q.e. } x \in \mathbb{R}^d \quad (37)$$

by [19, Theorem 4.6]. Moreover, since the Markov process  $\mathbf{M}^{\lambda^+\mu,h}$  has the transition density function

$$\frac{p^{\lambda^+\mu}(t, x, y)}{h(x)h(y)}$$

with respect to  $h^2m$ , (37) holds for all  $x \in \mathbb{R}^d$  by [19, Problem 4.6.3]. Using Lemma 6.2, (37), and [35, Chapter X, Proposition (3.11)], we see that  $\mathbf{M}^{\lambda^+\mu,h}$  is Harris recurrent.  $\square$

**Theorem 6.4.** *There exist a positive function  $g \in L^1(h^2m)$  and a function  $\psi \in C_0(\mathbb{R}^d)$  with  $\int_{\mathbb{R}^d} \psi h^2 dx = 1$  such that*

$$\int_{\mathbb{R}^d} |u(x) - h(x)L\left(\frac{u}{h}\right)|g(x)h(x)dx \leq C\mathcal{E}^{\lambda^+\mu}(u, u)^{1/2}, \quad u \in \mathcal{D}(\mathcal{E}^{\lambda^+\mu}), \quad (38)$$

where

$$L(u) = \int_{\mathbb{R}^d} u\psi h^2 dx.$$

*Proof.* By Proposition 6.3, we can apply Oshima's inequality in [29] to the Dirichlet form  $(\mathcal{E}^{\lambda^+\mu,h}, \mathcal{D}(\mathcal{E}^{\lambda^+\mu,h}))$  satisfying the Harris recurrence condition; there exist a positive function  $g \in L^1(h^2m)$  and a function  $\psi \in C_0(\mathbb{R}^d)$  with  $\int_{\mathbb{R}^d} \psi h^2 dx = 1$  such that

$$\int_{\mathbb{R}^d} |u(x) - L(u)|g(x)h^2(x)dx \leq C\mathcal{E}^{\lambda^+\mu,h}(u, u)^{1/2}, \quad u \in \mathcal{D}(\mathcal{E}^{\lambda^+\mu,h}) \quad (39)$$

where

$$L(u) = \int_{\mathbb{R}^d} u\psi h^2 dx.$$

Substituting  $v/h$  for  $u$  in (39) together with the equality

$$\mathcal{E}^{\lambda^+\mu,h}(v, v) = \mathcal{E}^{\lambda^+\mu}(hv, hv),$$

we obtain the equality (38).  $\square$

### 6.3 Proof of Differentiability of spectral functions

Before proving the differentiability of spectral function, we prepare a lemma relevant to general regular Dirichlet forms.

**Lemma 6.5.** *Let  $X$  be a locally compact separable metric space,  $m$  a positive Radon measure on  $X$ , and  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  a regular Dirichlet form on  $L^2(X; m)$ . Let  $\{u_n\} \subset \mathcal{D}(\mathcal{E})$  be a sequence with  $\lim_{n \rightarrow \infty} \mathcal{E}(u_n, u_n) = 0$  and  $\lim_{n \rightarrow \infty} u_n = 0$   $m$ -a.e. Then there is a subsequence  $\{u_{n_k}\}$  such that  $\lim_{k \rightarrow \infty} u_{n_k} = 0$  q.e.*

*Proof.* Let  $g$  be a non-negative continuous function with compact support and define

$$\mathcal{E}^g(u, u) = \mathcal{E}(u, u) + \int_X u^2 g dm. \quad (40)$$

Then  $(\mathcal{E}^g, \mathcal{D}(\mathcal{E}^g)(= \mathcal{D}(\mathcal{E})))$  becomes a transient Dirichlet form. Let  $u_n^{(l)} = ((-l) \vee u_n) \wedge l$ ,  $l = 1, 2, \dots$ . Then by the assumption  $\lim_{n \rightarrow \infty} \mathcal{E}^g(u_n^{(l)}, u_n^{(l)}) = 0$  for any  $l$ . Hence the 0-order version of [19, Theorem 2.1.4] says that there exists a subsequence  $\{u_{n_k}^{(l)}\}$  of  $\{u_n^{(l)}\}$  such that  $\lim_{k \rightarrow \infty} u_{n_k}^{(l)} = 0$   $\text{Cap}^{g, (0)}$ -q.e. Here  $\text{Cap}^{\mathcal{E}, (r)}$  denotes the  $r$ -order capacity with respect to  $(\mathcal{E}^g, \mathcal{D}(\mathcal{E}^g))$ . Note that by [19, Theorem 2.1.6],  $\text{Cap}^{g, (0)}$ -q.e. is equivalent to  $\text{Cap}^{g, (1)}$ -q.e. and  $\text{Cap}^{g, (1)}$ -q.e. is equivalent to  $\text{Cap}^{(1)}$ -q.e. because  $\mathcal{E}_1(u, u) \leq \mathcal{E}_1^g(u, u) \leq (1 + \|g\|_\infty) \mathcal{E}_1(u, u)$ .

Therefore we see that  $\lim_{k \rightarrow \infty} u_{n_k}^{(l)} = 0$  q.e. This proves the lemma because  $l$  is arbitrary.  $\square$

Next lemma implies that if  $C(\lambda) > 0$ ,  $C(\lambda)$  is eigenvalue of  $-\mathcal{H}^{\lambda\mu}$ .

**Lemma 6.6** ([48, Lemma 4.3]). *Let  $\mu = \mu^+ - \mu^- \in \mathcal{K}_d^\infty - \mathcal{K}_d^\infty$ . Then for any  $\lambda > \lambda^+$  and  $\lambda < \lambda^-$ , the negative spectrum of  $\sigma(\mathcal{E}^{\lambda\mu})$  consists of isolated eigenvalues with finite multiplicities.*

*Proof.* For  $\beta > 0, \epsilon > 0$  and  $\gamma > 0$ , let

$$\mathcal{E}^{(1)}(u, u) = \mathcal{E}^{\lambda^+ \mu^+}(u, u) - \beta \int_{\mathbb{R}^d} u^2 d\mu_{R^c}^- - (\lambda^+ - \epsilon) \int_{\mathbb{R}^d} u^2 d\mu^- + \gamma \int_{\mathbb{R}^d} u^2 dx.$$

Taking a constant  $\gamma' > 0$  so small that

$$\frac{\lambda^+ - \epsilon}{1 - \gamma'} \leq \lambda^+, \quad \gamma' < \frac{\gamma}{2}.$$

By Theorem 3.5,

$$\beta \int_{\mathbb{R}^d} u^2 d\mu_{R^c}^- \leq \gamma' \left( \mathcal{E}^{\lambda^+ \mu^+}(u, u) + \int_{\mathbb{R}^d} u^2 dx \right).$$

Since  $\beta \int_{\mathbb{R}^d} u^2 d\mu_{R^c}^-$  is relative compact form with respect to  $\mathcal{E}^{(1)}$ , the spectrum of  $\mathcal{E}^{\lambda^+ \mu^+}(u, u) - (\lambda^+ + \beta - \epsilon) \int_{\mathbb{R}^d} u^2 d\mu^-$  smaller than  $\inf\{\mathcal{E}^{(1)}(u, u)\} - \gamma$  consists of isolated eigenvalue with finite multiplicities by Lemma 1 in [27, 2.5.4]. Noting that

$$\begin{aligned} \mathcal{E}^{(1)}(u, u) &\geq \mathcal{E}^{\lambda^+ \mu^+}(u, u) - \gamma' \left( \mathcal{E}^{\lambda^+ \mu^+}(u, u) + \int_{\mathbb{R}^d} u^2 dx \right) - (\lambda^+ - \epsilon) \int_{\mathbb{R}^d} u^2 d\mu^- + \gamma \int_{\mathbb{R}^d} u^2 dx \\ &\geq (1 - \gamma') \left( \mathcal{E}^{\lambda^+ \mu^+}(u, u) - \lambda^+ \int_{\mathbb{R}^d} u^2 d\mu^- \right) + \frac{\gamma}{2} \int_{\mathbb{R}^d} u^2 dx \\ &\geq \frac{\gamma}{2} \int_{\mathbb{R}^d} u^2 dx, \end{aligned}$$



we see that  $\inf(\sigma(\mathcal{E}^{(1)})) \geq \frac{\gamma}{2}$ . Hence any negative sepctrum of  $\mathcal{E}^{\lambda^+\mu^+}(u, u) - (\lambda^+ + \beta - \epsilon) \int_{\mathbb{R}^d} u^2 d\mu^-$  is discrete because  $\gamma$  is arbitrary. Since  $\beta$  and  $\epsilon$  is arbitrary, we attain the lemma.  $\square$

**Theorem 6.7.** *Let  $\mu = \mu^+ - \mu^- \in \mathcal{S}_\infty - \mathcal{S}_\infty$ . If  $\mathcal{H}^{\lambda^+\mu}$  is null critical, then the spectral function  $C(\lambda)$  is differentiable.*

*Proof.* We deal with the case of  $\lambda \geq 0$ . First note that for  $\lambda > \lambda^+$ ,  $-C(\lambda)$  is the principal eigenvalue of the operator  $\mathcal{H}^{\lambda\mu} = \mathcal{H} - \lambda\mu$  by Lemma 6.6 and thus  $C(\lambda)$  is differentiable by the analytic perturbation theory [24, Chapter VII]. Hence it is enough to prove the differentiability of  $C(\lambda)$  at  $\lambda = \lambda^+$ . Furthermore, since  $C(\lambda)$  is convex by the definition, we have only to prove the existence of a sequence  $\{\lambda_n\}$  such that  $dC(\lambda_n)/d\lambda \downarrow 0$  as  $\lambda_n \downarrow \lambda^+$ .

By [24, p.405, Chapter VII (4.44)], we see

$$\frac{dC(\lambda)}{d\lambda} = - \int_{\mathbb{R}^d} u_\lambda^2 d\mu > 0, \quad \lambda > \lambda^+, \quad (41)$$

where  $u_\lambda$  is the  $L^2$ -normalized eigenfunction corresponding to the eigenvalue  $-C(\lambda)$ , that is,

$$-C(\lambda) = \mathcal{E}^{\lambda\mu}(u_\lambda, u_\lambda) = \lambda \int_{\mathbb{R}^d} u_\lambda^2 d\mu + \mathcal{E}(u_\lambda, u_\lambda). \quad (42)$$

Neglecting the positive part  $\mu^+$  of  $\mu$  in the (42), we have

$$\mathcal{E}(u_\lambda, u_\lambda) \leq -C(\lambda) + \lambda \int_{\mathbb{R}^d} u_\lambda^2 d\mu^-.$$

Furthermore, it follows form (12) that the right hand side above is dominated by

$$-C(\lambda) + \lambda\epsilon\mathcal{E}(u_\lambda, u_\lambda) + \lambda M(\epsilon).$$

Let  $\{\lambda_n\}$  be a sequence with  $\lim_{n \rightarrow \infty} \lambda_n \downarrow \lambda^+$ . Substituting  $\lambda_n$  for  $\lambda$  in the equation above and taking  $\epsilon > 0$  so small that  $\lambda_n\epsilon < 1$ , we have

$$\mathcal{E}(u_{\lambda_n}, u_{\lambda_n}) \leq \frac{-C(\lambda_n) + \lambda_n M(\epsilon)}{1 - \lambda_n\epsilon},$$

and thus

$$\limsup_{n \rightarrow \infty} \mathcal{E}(u_{\lambda_n}, u_{\lambda_n}) = \frac{\lambda^+ M(\epsilon)}{1 - \lambda^+\epsilon} < \infty \quad (43)$$

because  $C(\lambda_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Since by (43)

$$\begin{aligned} |\mathcal{E}^{\lambda^+\mu}(u_{\lambda_n}, u_{\lambda_n}) + C(\lambda_n)| &= |\mathcal{E}^{\lambda^+\mu}(u_{\lambda_n}, u_{\lambda_n}) - \mathcal{E}^{\lambda_n\mu}(u_{\lambda_n}, u_{\lambda_n})| \\ &\leq (\lambda_n - \lambda^+) \int_{\mathbb{R}^d} u_{\lambda_n}^2 d\mu \\ &\leq (\lambda_n - \lambda^+) (\|G|\mu|\|_\infty \mathcal{E}(u_{\lambda_n}, u_{\lambda_n})) \longrightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ ,

$$\lim_{n \rightarrow \infty} \mathcal{E}^{\lambda^+ \mu}(u_{\lambda_n}, u_{\lambda_n}) = 0. \quad (44)$$

Let  $h$  is the  $\mathcal{H}^{\lambda^+ \mu}$ -harmonic function constructed in Section 5 and  $(\mathcal{E}^{\lambda^+ \mu, h}, \mathcal{D}(\mathcal{E}^{\lambda^+ \mu, h}))$  the Dirichlet form generated by the  $h$ -transformed process. Then the equation (44) proves

$$\lim_{n \rightarrow \infty} \mathcal{E}^{\lambda^+ \mu, h}\left(\frac{u_{\lambda_n}}{h}, \frac{u_{\lambda_n}}{h}\right) = 0. \quad (45)$$

Let  $\psi$  and  $L(u)$  be the things in Theorem 38. Then since

$$\begin{aligned} \left|L\left(\frac{u_{\lambda_n}}{h}\right)\right| &= \int_{\mathbb{R}^d} u_{\lambda_n}(x) \psi(x) h(x) dx \\ &\leq \sqrt{\int_{\mathbb{R}^d} u_{\lambda_n}^2 dx} \sqrt{\int_{\mathbb{R}^d} \psi^2(x) h^2(x) dx} < \infty, \end{aligned}$$

we may assume that  $L(u_{\lambda_n}/h)$  converges to a certain constant  $C$  by taking a subsequence of  $\{\lambda_n\}$  if necessary. In addition, since 6.4 says

$$\begin{aligned} \int_{\mathbb{R}^d} |u_{\lambda_n} - Ch|gh dx &\leq \int_{\mathbb{R}^d} |u_{\lambda_n} - hL\left(\frac{u_{\lambda_n}}{h}\right)|gh dx + \int_{\mathbb{R}^d} |hL\left(\frac{u_{\lambda_n}}{h}\right) - Ch|gh dx \\ &\leq C\mathcal{E}^{\lambda^+ \mu}(u_{\lambda_n}, u_{\lambda_n})^{1/2} + \int_{\mathbb{R}^d} |L\left(\frac{u_{\lambda_n}}{h}\right) - C|gh^2 dx \rightarrow 0, \end{aligned}$$

we may assume that  $u_{\lambda_n} \rightarrow Ch$   $m$ -a.e. Now recall that  $\mathcal{H}^{\lambda^+ \mu}$  is null critical if and only if  $d \leq 2\alpha$ . Then the constant  $C$  must be equal to 0 because

$$1 = \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} u_{\lambda_n}^2 dx \geq \int_{\mathbb{R}^d} \liminf_{n \rightarrow \infty} u_{\lambda_n}^2 dx = C^2 \int_{\mathbb{R}^d} h^2 dx,$$

and consequently

$$\lim_{n \rightarrow \infty} u_{\lambda_n} = 0, \quad m\text{-a.e.} \quad (46)$$

Notice that  $\mathcal{E}^{\lambda^+ \mu, h}$ -q.e. is equivalent to  $\mathcal{E}$ -q.e. Then combing (45) and (46) with lemma 6.5, we may assume that  $u_{\lambda_n}$  converges to 0 q.e.

Since  $u_{\lambda_n}$  is the eigenfunction corresponding to  $C(\lambda_n)$ ,

$$u_{\lambda_n} = e^{-C(\lambda_n)t} P_t^{\lambda_n \mu} u_{\lambda_n}.$$

and

$$\|u_{\lambda_n}\|_{\infty} \leq e^{-C(\lambda_n)t} \|P_t^{-\lambda_n \mu^-}\|_{2, \infty} \leq \|P_t^{-\lambda_1 \mu^-}\|_{2, \infty} < \infty$$

by [2, Theorem 6.1 (iii)]. Hence we have

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}^d} u_{\lambda_n}^2 d\mu \right| &\leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^d} u_{\lambda_n}^2 d|\mu| \\
&= \limsup_{n \rightarrow \infty} \left( \int_{\mathbb{R}^d} u_{\lambda_n}^2 d|\mu|_R + \int_{\mathbb{R}^d} u_{\lambda_n}^2 d|\mu|_{R^c} \right) \\
&\leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^d} u_{\lambda_n}^2 d|\mu|_R + \limsup_{n \rightarrow \infty} \|G|\mu|_{R^c}\|_{\infty} \mathcal{E}(u_{\lambda_n}, u_{\lambda_n}) \\
&\leq \|G|\mu|_{R^c}\|_{\infty} \frac{\lambda^+ M(\epsilon)}{1 - \lambda^+ \epsilon}.
\end{aligned}$$

By letting  $R$  to  $\infty$ , we complete the proof.  $\square$

**Remark 6.8.** *In the case that the symmetric Markov process  $\mathbf{M}$  is recurrent, for example 1 or 2-dimensional Brownian motion, symmetric  $\alpha$ -stable process ( $d \leq \alpha$ ) and 1 or 2-dimensional relativistic  $\alpha$ -stable process and  $\mu = -\mu \in \mathcal{K}_d^{\infty}$ , we can prove the differentiability of spectral function by the exactly same as in [49].*

## 6.4 Non-differentiability of spectral functions

When  $\mathbf{M}$  is the Brownian motion and the potential  $\mu$  is absolutely continuous with respect to the Lebesgue measure, non-differentiability of spectral function was considered in [41]. The argument in [41, Theorem 2.1] can be adapted to prove non-differentiability of spectral function.

**Theorem 6.9.** *If  $\mathcal{H}^{\lambda^+ \mu}$  is positive critical, then  $C(\lambda)$  is not differentiable.*

*Proof.* Note that the ground state  $h$  belongs to  $L^2(m)$ , that is, zero is an eigenvalue of  $\mathcal{H}^{\lambda^+ \mu}$ . We normalize the function  $h$  as  $\|h\|_2 = 1$ . Let  $\{u_{\lambda_n}\}$  be the sequence defined in the proof of Theorem 6.7, that is,  $u_{\lambda_n}$  is the  $L^2(m)$ -normalized eigenfunction corresponding with the eigenvalue  $\lambda_n$  ( $\lambda_n > \lambda^+$ ). Since  $\{u_{\lambda_n}\}$  is bounded in  $\mathcal{E}^{(\alpha)}$  and in  $L^2(m)$ , we may suppose that

$$u_{\lambda_n} \rightarrow u_0, \quad \text{weakly in } \mathcal{E} \text{ and in } L^2(m).$$

Moreover we know in the proof of Theorem 6.7 that

$$u_{\lambda_n} \rightarrow Ch, \quad m\text{-a.e.}$$

Hence  $u_0 = Ch$ ,  $m$ -a.e., and thus the constant  $C$  is less than or equal to 1. Since for  $\lambda > \lambda^+$ ,

$$-C(\lambda) \leq \mathcal{E}(h, h) + \lambda \int h^2 d\mu$$

and

$$\mathcal{E}(h, h) = -\lambda^+ \int_{\mathbb{R}^d} h^2 d\mu,$$

we have

$$\frac{C(\lambda)}{\lambda - \lambda^+} \geq - \int h^2 d\mu. \quad (47)$$

Noting that there exists a constant  $\theta \in (0, 1)$  such that

$$\frac{C(\lambda)}{\lambda - \lambda^+} = C'(\lambda^+ + \theta(\lambda - \lambda^+)).$$

by the mean value theorem, we can find a sequence  $\{a_n\}$  such that  $a_n \rightarrow \lambda^+$  as  $n \rightarrow \infty$  and

$$\limsup_{\lambda \rightarrow \lambda^+} \frac{C(\lambda)}{\lambda - \lambda^+} = \lim_{n \rightarrow \infty} C'(a_n).$$

By Assumption 6.1,

$$\lim_{n \rightarrow \infty} C'(a_n) = - \lim_{n \rightarrow \infty} \int u_{a_n}^2 d\mu = -C^2 \int_{\mathbb{R}^d} h^2 d\mu.$$

Since  $h > 0 \in \mathcal{D}_e(\mathcal{E})$  and  $(\mathcal{E}, \mathcal{D}_e(\mathcal{E}))$  is Hilbert space, we know that

$$- \int_{\mathbb{R}^d} h^2 d\mu = \frac{1}{\lambda^+} \mathcal{E}(h, h) > 0.$$

Hence

$$\limsup_{\lambda \rightarrow \lambda^+} \frac{C(\lambda)}{\lambda - \lambda^+} \leq -C^2 \int_{\mathbb{R}^d} h^2 d\mu \leq - \int_{\mathbb{R}^d} h^2 d\mu.$$

On the other hand, by (47), we find that

$$\liminf_{\lambda \rightarrow \lambda^+} \frac{C(\lambda)}{\lambda - \lambda^+} \geq - \int_{\mathbb{R}^d} h^2 d\mu.$$

Therefore

$$\lim_{\lambda \rightarrow \lambda^+} \frac{C(\lambda)}{\lambda - \lambda^+} = - \int_{\mathbb{R}^d} h^2 d\mu > 0.$$

□

## 7 Large deviation principle for additive functionals

The large deviation principle (LDP) for additive functionals is our motivation to prove the differentiability of spectral functions.

Now we introduce the Gärtner-Ellis theorem. This part is due to [17]. We modified to the continuous time. We put

$$\Lambda_t(\lambda) := \log E_x \left[ \exp \left( -\lambda \frac{A_t^\mu}{t} \right) \right].$$

### Assumption

For each  $\lambda \in \mathbb{R}$ , the *logarithmic moment generating function*, defined as the limit

$$\Lambda(\lambda) := \lim_{t \rightarrow \infty} \frac{1}{t} \Lambda_t(t\lambda)$$

exists as an extended real number. Further, the origin belongs to the interior of  $\mathcal{D}_\Lambda := \{\lambda \in \mathbb{R} : \Lambda(\lambda) < \infty\}$ .

**Definition 7.1.** A convex function  $\Lambda : \mathbb{R} \rightarrow (-\infty, \infty]$  is essentially smooth if:

- (I)  $\mathcal{D}_\Lambda^\circ$  is non-empty.
- (II)  $\Lambda(\cdot)$  is differentiable throughout  $\mathcal{D}_\Lambda^\circ$ .
- (III)  $\Lambda(\cdot)$  is steep, namely,  $\lim_{n \rightarrow \infty} |\Lambda'(\lambda_n)| = \infty$  whenever  $\{\lambda_n\}$  is a sequence in  $\mathcal{D}_\Lambda^\circ$  converging to a boundary point of  $\mathcal{D}_\Lambda^\circ$ .

If the spectral function becomes the logarithmic moment generating function, the differentiability of spectral function claims essential smoothness because  $\Lambda(\lambda)$  is convex and finite for all  $\lambda \in \mathbb{R}$ .

**Theorem 7.2** (Gärtner-Ellis theorem Theorem 2.3.6 [17]). *The following estimate follows;*

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \log P_x \left( \frac{A_t^\mu}{t} \in G \right) &\geq - \inf_{\theta \in G} I(\theta) \\ \limsup_{t \rightarrow \infty} \frac{1}{t} \log P_x \left( \frac{A_t^\mu}{t} \in F \right) &\leq - \inf_{\theta \in F} I(\theta), \end{aligned}$$

where  $I(\theta)$  is the Legendre transform of  $C(\lambda)$ , that is,

$$I(\theta) = \sup_{\lambda \in \mathbb{R}} \{\theta\lambda - C(\lambda)\}.$$

and  $G$  and  $F$  are open and closed set in  $\mathbb{R}$  respectively.

This approach, that is, the differentiability of spectral function, may be unique because it may be difficult to prove the differentiability of it.

Now the differentiability of logarithmic moment generating function is just sufficient condition and not necessary condition for the LDP of additive functionals. There exist an example that the LDP holds unless the logarithmic moment generating function is differentiable. In §9, we will introduce it.

## 8 Examples

In this section, we would like to introduce some examples of symmetric Markov processes which hold differentiability of spectral functions. In the sequel, the function  $h$  denote the ground state of the critical Schrödinger operator  $\mathcal{H}^{\lambda+\mu}$ . In these examples, using the

method in §4 and §5, we can construct the finely continuous and bounded ground state for critical Schrödinger operator  $\mathcal{H}^{\lambda+\mu}$ . We have not introduced ways to prove the continuity of  $h$  yet. There are differences among these processes each other.

The first example ([51]) is that the Markov process is the standard Brownian motion. Since the Brownian motion has a continuous path with respect to times, we can apply the local property. So in this case, using the local property of  $\mathcal{H}^{\lambda+\mu}$ , we can prove the continuity of  $h$ . Also using the well-known Harnack inequality of harmonic functions, we obtain the order of decay of  $h$  at infinity.

The second example ([52]) is that the Markov process is the symmetric  $\alpha$ -stable process. This process is a pure jump process. Since this process is non-local, we can not apply the similar method ([51]) of the Brownian case to prove the continuity of  $h$ . Recently, the Harnack inequality is studied for non-local type operator. By lemma 5.14 and [5] and the Harnack inequality, we can obtain the continuity of  $h$ . Moreover there is an alternative proof of the continuity of  $h$ , we will introduce it. Furthermore we also obtain the order of decay of  $h$  by the Harnack inequality.

The third example ([54]) is that the Markov process is the relativistic  $\alpha$ -stable process. In this case, the process is also non-local. Readers may think that it can be used the method for symmetric  $\alpha$ -stable processes. We would like to emphasis to use the explicit form of the Green function to prove the continuity of  $h$  for symmetric  $\alpha$ -stable processes. Since we do not know the explicit form of the Green function, we can not apply the same argument for symmetric  $\alpha$ -stable processes. Hence we develop a new method to prove the continuity. If  $d$  is greater than 2, the process is transient. So there exists a minimal Green function of this process. We note that the function  $h$  is bounded and  $P_t^\mu$  has strong Feller property when  $\mu \in \mathcal{K}_d^\infty$ . Since the function  $h$  is bounded and  $P_t^{\lambda+\mu}$ -invariant, we can prove the continuity of  $h$ .

## 8.1 Brownian motions

In this subsection, we refer to the paper [51]. If  $p(\xi) = \frac{1}{2}|\xi|^2$ , the symmetric Markov process  $\mathbf{M}$  is called Brownian motion. The generator of this process is denoted by  $\mathcal{H} = \frac{1}{2}\Delta$ . In this case, we can extend the class of measure  $\mathcal{S}_\infty$  to  $\mathcal{K}_d^\infty$  since it is known that  $\mathcal{S}_\infty$  is identical to  $\mathcal{K}_d^\infty$ . Here we only consider the case that  $\mu = -\mu \in \mathcal{K}_d^\infty$  here.

It is well-known that if  $d \geq 3$ , this is transient and if  $d \leq 2$ , this is recurrent. If  $d \geq 3$ , the Green function is

$$G(x, y) = c(d)|x - y|^{2-d}.$$

The Dirichlet form is

$$\begin{aligned} \mathcal{E}^{(2)}(u, v) &= \frac{1}{2}\mathbf{D}(u, v) \\ \mathcal{D}(\mathcal{E}^{(2)}) &= H^1(\mathbb{R}^d), \end{aligned}$$

where  $\mathbf{D}$  denotes the classical Dirichlet integral and  $H^1(\mathbb{R}^d)$  is the Sobolev space of order 1 ([19, Example 4.4.1]).

This case is local, so we can prove regularity of ground state using the well-known method.

**Lemma 8.1.** For  $\mu \in \mathcal{K}_d^\infty$ , there exists a positive continuous function such that  $\mathcal{H}^{\lambda^+ \mu} h = 0$ .

*Proof.* Let  $\lambda_n$  be the bottom of spectrum of  $\mathcal{H}^{\lambda^+ \mu}$  for the Dirichlet problem on  $B(n)$ . Since  $0 = -C(\lambda^+) < \lambda_{n+1} < \lambda_n$ ,  $\mathcal{H}^{\lambda^+ \mu}$  is subcritical on  $B(n)$ . Let  $G^n$  denotes the Green operator of  $\mathcal{H}^{\lambda^+ \mu}$  on  $B(n)$ . We define a function  $h_n$  by  $h_n(x) = c_n G^{n+1} I_{A_n}(x)$ , where  $I_{A_n}$  is the indicator function of  $A_n (= B(n+1) \setminus B(n))$  and  $c_n$  is the normalized constant,  $c_n = (G^{n+1} I_{A_n}(0))^{-1}$ . Then  $h_n$  is a harmonic function on  $B(m)$ ,  $m < n$ . Indeed, for  $x \in B(m)$

$$\begin{aligned} E_x[\exp(\lambda^+ A_{\tau_m}^\mu) h_n(B_{\tau_m})] &= c_n E_x[\exp(\lambda^+ A_{\tau_m}^\mu) G^{m+1} I_{A_n}(B_{\tau_m})] \\ &= c_n E_x \left[ \exp(\lambda^+ A_{\tau_m}^\mu) E_{B_{\tau_m}} \left[ \int_0^{\tau_{n+1}} \exp(\lambda^+ A_t^\mu) I_{A_n}(B_t) dt \right] \right], \end{aligned}$$

where  $\tau_m = \inf\{t > 0 : B_t \not\subset B(m)\}$ . By the strong Markov property, the right hand side is equal to

$$\begin{aligned} c_n E_x \left[ \int_0^{\tau_{n+1} \circ \theta_{\tau_m}} \exp(\lambda^+ (A_{\tau_m}^\mu + A_t^\mu \circ \theta_{\tau_m})) I_{A_n}(B_{t+\tau_m}) dt \right] \\ = c_n E_x \left[ \int_{\tau_m}^{\tau_{n+1} \circ \theta_{\tau_m} + \tau_m} \exp(\lambda^+ A_t^\mu) I_{A_n}(B_t) dt \right]. \end{aligned}$$

Noting that  $\tau_{n+1} \circ \theta_{\tau_m} + \tau_m = \tau_{n+1}$  and  $\int_0^{\tau_m} \exp(\lambda^+ A_t^\mu) I_{A_n}(B_t) dt = 0$ , we see that the last term is equal to  $h_n(x)$ . Therefore  $h_n$  satisfies (22) for  $G = B(m)$ .

Now by [6, Corollary 7.8],  $\{h_n\}$  is uniformly bounded and equicontinuous on  $B(1)$ , so we can choose a subsequence of  $\{h_n\}$  which converges uniformly on  $B(1)$ . We denote the subsequence by  $\{h_n^{(1)}\}$ . Next take a subsequence  $\{h_n^{(2)}\}$  of  $\{h_n^{(1)}\}$  so that it converges uniformly on  $B(2)$ . By the same procedure, we take a subsequence  $\{h_n^{(m+1)}\}$  of  $\{h_n^{(m)}\}$  so that it converges uniformly on  $B(m+1)$ . Then the function,  $h(x) = \lim_{n \rightarrow \infty} h_n^{(n)}(x)$ , is a desired one.  $\square$

**Lemma 8.2.** Let  $\mu \in \mathcal{K}_d^\infty$ . Then the number  $\lambda^+$  is characterized as a unique positive number such that

$$\inf \left\{ \frac{1}{2} \mathbf{D}(u, u) : \lambda^+ \int_{\mathbb{R}^d} u^2 d\mu = 1 \right\} = 1. \quad (48)$$

*Proof.* Define

$$F(\lambda) = \inf \left\{ \frac{1}{2} \mathbf{D}(u, u) : \lambda \int_{\mathbb{R}^d} u^2(x) \mu(dx) = 1 \right\},$$

Note that  $F(\lambda) = F(1)/\lambda$ . Then  $F(1)$  is nothing but the bottom of spectrum of the time changed process by the additive functional  $A_t^\mu$  ([46, Lemma 3.1]). We see by [47, Lemma 3.1] that 1-resolvent  $R_1^\mu$  of the time changed process satisfies  $R_1^\mu 1 \in C_\infty(\mathbb{R}^d)$ . Hence it follows from [37, Corollary 3.2] and [47, Corollary 2.2] that  $F(1) > 0$ . Consequently we see that  $\lambda^0 = F(1)$  is a unique positive constant such that  $F(\lambda^0) = 1$ . Lemma 3.2 leads us that  $\lambda^0 = \lambda^+$ .  $\square$

**Corollary 8.3.** For  $\mu \in \mathcal{K}_d^\infty$ , the operator  $\mathcal{H}^{\lambda+\mu}$  is critical.

*Proof.* Let  $F(\lambda)$  be the function in the proof of Lemma 8.2. Then it is known in [49, Theorem 3.9] that the operator  $\mathcal{H}^{\lambda\mu}$  is subcritical if and only if  $F(\lambda) > 1$ . Hence by Lemma 8.1 and Lemma 8.2,  $\mathcal{H}^{\lambda+\mu}$  is critical.  $\square$

**Lemma 8.4.** A positive  $\mathcal{H}^{\lambda+\mu}$ -harmonic function  $h$  satisfies  $P_t^{\lambda+\mu}h(x) \leq h(x)$ .

*Proof.* Let  $x \in B(m)$ . By Definition 3.1,  $h$  satisfies

$$h(x) = E_x[\exp(\lambda^+ A_{\tau_n}^\mu)h(B_{\tau_n})]$$

for any  $n > m$ . Here  $\tau_n$  is the first exit time from  $B(n)$ . It follows from the Markov property that

$$\begin{aligned} & E_x[\exp(\lambda^+ A_t^\mu)h(B_t); t < \tau_m] \\ &= E_x[\exp(\lambda^+ A_t^\mu) \exp(\lambda^+ A_{\tau_n}^\mu \circ \theta_t)h(B_{\tau_n} \circ \theta_t); t < \tau_m] \\ &= E_x[\exp(\lambda^+ A_{\tau_n}^\mu)h(B_{\tau_n}); t < \tau_m] \leq h(x). \end{aligned}$$

Hence we have

$$P_t^{\lambda+\mu}h(x) = \lim_{m \rightarrow \infty} E_x[\exp(\lambda^+ A_t^\mu)h(B_t); t < \tau_m] \leq h(x).$$

$\square$

In this case, for  $\mu \in \mathcal{K}_d^\infty$  the assumption 3.9 is always satisfied.

**Lemma 8.5.** If  $\mu \in \mathcal{K}_d^\infty$ , then the embedding of  $H_e^1(\mathbb{R}^d)$  to  $L^2(\mu)$  is compact.

*Proof.* Let  $\{u_n\}$  be a sequence in  $H_e^1(\mathbb{R}^d)$  such that

$$u_n \rightarrow u_0 \in H_e^1(\mathbb{R}^d), \quad \mathbf{D}\text{-weakly.}$$

Rellich's theorem says that for any compact set  $K \subset \mathbb{R}^d$

$$u_n I_K \rightarrow u_0 I_K \quad L^2(m)\text{-strongly.} \quad (49)$$

Now, for  $\varphi \in C_0^\infty(\mathbb{R}^d)$  with  $\varphi = 1$  on  $B(R)$

$$\begin{aligned} \int_{\mathbb{R}^d} |u_n - u_0|^2 \mu_R(dx) &= \int_{\mathbb{R}^d} |u_n \varphi - u_0 \varphi|^2 \mu_R(dx) \\ &\leq \epsilon \mathbf{D}(u_n \varphi - u_0 \varphi, u_n \varphi - u_0 \varphi) + M(\epsilon) \int_{\mathbb{R}^d} |u_n \varphi - u_0 \varphi|^2 dx \end{aligned}$$

by (12), and the second term converges to 0 as  $n \rightarrow \infty$  by (49). Since

$$\sup_n \mathbf{D}(u_n \varphi - u_0 \varphi, u_n \varphi - u_0 \varphi) < \infty$$



by the principle of uniform boundedness and  $\epsilon$  is arbitrary,  $u_n$  converges to  $u_0$  in  $L^2(\mu_R)$ . Moreover, since by Theorem 3.5,

$$\begin{aligned} \int_{\mathbb{R}^d} |u_n - u_0|^2 \mu(dx) &= \int_{\mathbb{R}^d} |u_n - u_0|^2 \mu_R(dx) + \int_{\mathbb{R}^d} |u_n - u_0|^2 \mu_{R^c}(dx) \\ &\leq \int_{\mathbb{R}^d} |u_n - u_0|^2 \mu_R(dx) + \|G\mu_{R^c}\|_\infty \mathbf{D}(u_n - u_0, u_n - u_0), \\ \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^d} |u_n - u_0|^2 \mu(dx) &\leq \|G\mu_{R^c}\|_\infty \sup_n \mathbf{D}(u_n - u_0, u_n - u_0). \end{aligned}$$

Hence according to the definition of  $\mathcal{K}_d^\infty$  the right hand side converges to 0 by letting  $R$  to  $\infty$ . Therefore  $\{u_n\}$  is an  $L^2(\mu)$ -convergent sequence.  $\square$

Now we consider asymptotic of  $h$  as  $|x| \rightarrow \infty$ . Let  $w$  be a positive continuous function with compact support. Suppose that  $0 \in \text{supp}[w] \subset B(R)$ . By Theorem 5.11 and the continuity of  $h$

$$c \int_{B(R)} G^\nu(x, y) w(y) dy \leq h(x) \leq C \int_{B(R)} G^\nu(x, y) w(y) dy,$$

and so by the inequality (24),

$$c \int_{B(R)} G(x, y) w(y) dy \leq h(x) \leq C \int_{B(R)} G(x, y) w(y) dy.$$

The Harnack inequality to  $\{G(x, \cdot)\}_{\{x \in B(R)^c\}}$  says that for any  $x \in B(R)^c$  and  $y \in \text{supp}[w]$

$$cG(x, y) \leq G(x, 0) \leq CG(x, y).$$

Therefore we see that

$$cG(x, 0) \leq h(x) \leq CG(x, 0) \quad \text{for } x \in B(R)^c,$$

namely,

$$\frac{c}{|x|^{d-2}} \leq h(x) \leq \frac{C}{|x|^{d-2}} \quad \text{for } x \in B(R)^c. \quad (50)$$

Hence we obtain the following theorem.

**Theorem 8.6.** *The operator  $\mathcal{H}^{\lambda+\mu}$  is null ciritical if and only if  $d \leq 4$ .*

Therefore if  $d \leq 4$ , we obtain the differentiability of spectral functions.

Finally in this subsection, we introduce the concrete example.

**Example 8.7** ([49, Example 3.1]). *Let  $d = 1$  and  $\mu(dx) = \delta_0 \in \mathcal{K}_1^\infty$ . It is known that*

$$C(\lambda) = \begin{cases} \frac{\lambda^2}{2} & \lambda \geq 0 \\ 0 & \lambda < 0. \end{cases}$$

Hence the Legendre transform of  $C(\lambda)$  is

$$I(\theta) = \begin{cases} \frac{\theta^2}{2} & \theta \geq 0 \\ \infty & \theta < 0. \end{cases}$$

## 8.2 Symmetric $\alpha$ -stable processes

In this subsection, we refer to the paper [52]. If  $p(\xi)$  equals  $\frac{1}{2}|\xi|^\alpha$  ( $0 < \alpha \leq 2$ ), the symmetric Markov process  $\mathbf{M}$  is called the symmetric  $\alpha$ -stable process. The generator of this process is denoted by  $\mathcal{H} = -\frac{1}{2}(-\Delta)^{\alpha/2}$ . Recall that if  $\alpha = 2$ , the process is the Brownian motion. In this case, we can prove the continuity of  $h$  for  $\mu = \mu^+ - \mu^- \in \mathcal{K}_d^\infty - \mathcal{K}_d^\infty$ .

If  $\alpha < d$ , the process  $\mathbf{M}^\alpha$  is transient. If  $\alpha \geq d$ , the process  $\mathbf{M}^\alpha$  is recurrent. Let  $p(t, x, y)$  be the transition density function of  $\mathbf{M}^\alpha$ . If  $\alpha < d$ , the Green function is

$$G(x, y) = \int_0^\infty p(t, x, y) dt = C(d, \alpha) |x - y|^{\alpha-d},$$

where  $C(d, \alpha) = 2^{1-\alpha} \pi^{-d/2} \Gamma(\frac{d-\alpha}{2}) \Gamma(\frac{\alpha}{2})^{-1}$ .

When  $\mathbf{M}^\alpha$  is a symmetric  $\alpha$ -stable process in  $\mathbb{R}^d$  with  $d > \alpha$ , it is well-known that the following 3G-inequality holds:

$$\frac{G(x, y)G(y, z)}{G(x, z)} \leq C(G(x, y) + G(y, z)).$$

So we find that  $\mathcal{K}_d^\infty = \mathcal{S}_\infty$ . Therefore in this case, we can prove the differentiability of spectral function when  $\alpha < d \leq 2\alpha$  and  $\mu = \mu^+ - \mu^- \in \mathcal{K}_d^\infty - \mathcal{K}_d^\infty$ .

For a measure  $\mu$ , the 0-potential of  $\mu$  is defined by

$$G\mu(x) = \int_{\mathbb{R}^d} G(x, y) \mu(dy).$$

Let  $P_t$  be the semigroup of  $\mathbf{M}^\alpha$ ,

$$P_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y) dy = E_x[f(X_t)].$$

Let  $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$  be the Dirichlet form generated by  $\mathbf{M}^\alpha$ . For  $0 < \alpha < 2$ , it is given by

$$\begin{aligned} \mathcal{E}^{(\alpha)}(u, v) &= \frac{1}{2} \mathcal{A}(d, \alpha) \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} dx dy \\ \mathcal{D}(\mathcal{E}^{(\alpha)}) &= \left\{ u \in L^2(\mathbb{R}^d) : \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dx dy < \infty \right\}, \end{aligned}$$

where

$$\mathcal{A}(d, \alpha) = \frac{\alpha 2^{d-1} \Gamma(\frac{\alpha+d}{2})}{\pi^{d/2} \Gamma(1 - \frac{\alpha}{2})}$$

([19, Example 1.4.1]).

Let  $(\mathcal{E}^{(\alpha)}, \mathcal{D}_e(\mathcal{E}^{(\alpha)}))$  denote the extended Dirichlet form of  $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$  ([19, p.36]). Then  $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$  is a Hilbert space with inner product  $\mathcal{E}^{(\alpha)}$  because  $\mathbf{M}^\alpha$  is transient ([19, Theorem 1.5.3]).

For symmetric  $\alpha$ -stable processes, we can prove that for  $\mu \in \mathcal{K}_d^\infty$  the embedding of  $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$  to  $L^2(\mu)$  is compact. Therefore we can except (I) from Assumption 3.9. Let us prove it.

**Lemma 8.8.** *Let  $\varphi \in C_0^\infty(\mathbb{R}^d)$  and  $u \in \mathcal{D}_e(\mathcal{E}^{(\alpha)})$ . Then  $u\varphi \in \mathcal{D}_e(\mathcal{E}^{(\alpha)})$  and there exists a constant  $C$  depending only on  $\varphi$  such that*

$$\mathcal{E}^{(\alpha)}(u\varphi, u\varphi) \leq C\mathcal{E}^{(\alpha)}(u, u). \quad (51)$$

*Proof.* By the definition of  $\mathcal{E}^{(\alpha)}$  and the inequality “ $(a+b)^2 \leq 2(a^2+b^2)$ ”,

$$\begin{aligned} & \mathcal{E}^{(\alpha)}(u\varphi, u\varphi) \\ &= \frac{1}{2} \mathcal{A}(d, \alpha) \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \frac{(u(x)\varphi(x) - u(y)\varphi(y))^2}{|x-y|^{d+\alpha}} dx dy \\ &\leq \mathcal{A}(d, \alpha) \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \frac{u(x)^2(\varphi(x) - \varphi(y))^2}{|x-y|^{d+\alpha}} dx dy \\ &\quad + \mathcal{A}(d, \alpha) \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \frac{\varphi(y)^2(u(x) - u(y))^2}{|x-y|^{d+\alpha}} dx dy \\ &= \mathcal{A}(d, \alpha)((\text{I}) + (\text{II})). \end{aligned}$$

Since

$$(\text{II}) \leq \|\varphi\|_\infty^2 \mathcal{E}^{(\alpha)}(u, u), \quad (52)$$

we only consider the term (I).

Take  $R$  so large that  $\text{supp } \varphi \subset B(R-1)$ . Then (I) equals to

$$\begin{aligned} & \iint_{B(R) \times B(R) \setminus \Delta} \frac{u(x)^2(\varphi(x) - \varphi(y))^2}{|x-y|^{d+\alpha}} dx dy + \int_{B(R)} \left( \int_{B(R)^c} \frac{u(x)^2\varphi(x)^2}{|x-y|^{d+\alpha}} dy \right) dx \\ &+ \int_{B(R)^c} u(x)^2 \left( \int_{B(R-1)} \frac{\varphi(y)^2}{|x-y|^{d+\alpha}} dy \right) dx \\ &= (\text{III}) + (\text{IV}) + (\text{V}). \end{aligned}$$

Since  $|\varphi(x) - \varphi(y)|^2 \leq C|x-y|^2$ , we have

$$\begin{aligned} (\text{III}) &\leq \int_{B(R)} \left( \int_{B(R) \cap \{|x-y| \leq 1\}} + \int_{B(R) \cap \{|x-y| \geq 1\}} \frac{u(x)^2(\varphi(x) - \varphi(y))^2}{|x-y|^{d+\alpha}} dy \right) dx \\ &\leq C \int_{B(R)} u(x)^2 \left( \int_{B(R) \cap \{|x-y| \leq 1\}} \frac{dy}{|x-y|^{d+\alpha-2}} \right) dx \\ &\quad + C \int_{B(R)} u(x)^2 \left( \int_{B(R) \cap \{|x-y| \geq 1\}} \frac{dy}{|x-y|^{d+\alpha}} \right) dx \\ &\leq C \int_{B(R)} u(x)^2 \left( \int_0^1 r^{1-\alpha} dr \right) dx + C \int_{B(R)} u(x)^2 \left( \int_1^\infty r^{-(\alpha+1)} dr \right) dx \\ &\leq C \int_{B(R)} u(x)^2 dx. \end{aligned}$$

By Hölder's inequality the right hand side is less than

$$C \left( \int_{B(R)} u(x)^p dx \right)^{2/p} \left( \int_{B(R)} 1 dx \right)^{1/q} \leq C \left( \int_{B(R)} u(x)^p dx \right)^{2/p},$$

where  $p = \frac{2d}{d-\alpha}$  and  $2/p + 1/q = 1$ . Using Sobolev's inequality for order  $\alpha/2$  ([19, p.44, (1.5.20)]), we can see that (III) is dominated by  $C\mathcal{E}^{(\alpha)}(u, u)$ .

The term (IV) is dominated by

$$C \int_{B(R)} u(x)^2 \left( \int_R^\infty r^{-(1+\alpha)} dr \right) dx \leq C \int_{B(R)} u(x)^2 dx,$$

and by the same reason as above

$$(IV) \leq C\mathcal{E}^{(\alpha)}(u, u).$$

Finally, we will consider (V). Since  $|x - y| \geq 1$  on  $(x, y) \in B(R)^c \times B(R - 1)$ ,

$$(V) \leq \int_{B(R)^c} u(x)^2 \left( \int_{B(R-1)} \varphi(y)^2 dy \right) dx \leq C \int_{\mathbb{R}^d} u(x)^2 dx \leq C\mathcal{E}^{(\alpha)}(u, u).$$

Hence

$$(I) \leq C\mathcal{E}^{(\alpha)}(u, u). \quad (53)$$

□

**Lemma 8.9.** *Let  $u_n \in \mathcal{D}_e(\mathcal{E}^{(\alpha)})$ ,  $n = 1, 2, \dots$ , be a sequence such that  $u_n$  converges to  $u$  weakly in  $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$ . Then for any set  $A$  of finite Lebesgue measure,  $u_n I_A$  converges to  $u I_A$  strongly in  $L^2(m)$ .*

*Proof.* The proof of this lemma is just the argument in [26, Theorem 8.6].

First note that the semigroup  $P_t$  of  $\mathbf{M}^\alpha$  can be uniquely extended to a linear operator on  $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$  and that

$$\|u - P_t u\|_2 \leq \sqrt{t} \mathcal{E}^{(\alpha)}(u, u)^{1/2}, \quad u \in \mathcal{D}_e(\mathcal{E}^{(\alpha)})$$

(see [19, Lemma 1.5.4]). We then have

$$\begin{aligned} \|(u_n - u)I_A\|_2 &\leq \|(u_n - P_t u_n)I_A\|_2 + \|(P_t u_n - P_t u)I_A\|_2 + \|(P_t u - u)I_A\|_2 \\ &\leq 2\sqrt{t} \sup_n \mathcal{E}^{(\alpha)}(u_n, u_n) + \|(P_t u_n - P_t u)I_A\|_2. \end{aligned} \quad (54)$$

By the Sobolev inequality,  $u_n$  is a bounded sequence in  $L^p(m)$ ,  $1/p = 1/2 - \alpha/2d$  and thus there exists an  $L^p(m)$ -weakly convergent subsequence. Using the Banach-Saks Theorem, as in the proof of [19, Lemma 3.2.2], we can show that the entire sequence  $u_n$  converges to  $u$  weakly in  $L^p(m)$ . Using the Sobolev inequality again, we see that the integral kernel  $p_t(x, y)$  of  $P_t$  is bounded. Consequently,  $p_t(x, \cdot) \in L^q(m)$  ( $1/q + 1/p = 1$ ),  $\|P_t u_n\|_\infty$  is bounded in  $n$ , and  $P_t u_n$  converges to  $P_t u$   $m$ -a.e. Hence, by the dominated convergence theorem the last term of (54) converges to zero as  $n \rightarrow \infty$ . This lemma follows by letting  $n \rightarrow \infty$  and  $t \rightarrow 0$  in (54). □

For a measure  $\mu$ , let us denote

$$\mu_R(\cdot) = \mu(\cdot \cap B(R)), \quad \mu_{R^c} = \mu(\cdot \cap B(R)^c).$$

**Theorem 8.10.** *If  $\mu \in \mathcal{K}_d^\infty$ , then the embedding of  $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$  into  $L^2(\mu)$  is compact.*

*Proof.* First note that the embedding of  $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$  into  $L^2(\mu)$  is bounded by Theorem 3.5. Let  $\{u_n\}$  be a sequence in  $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$  such that  $u_n \rightarrow u$  weakly in  $\mathcal{D}_e(\mathcal{E}^{(\alpha)})$ . Then Lemma 8.9 says that for  $R > 0$

$$u_n I_{B(R)} \rightarrow u I_{B(R)} \quad L^2(m)\text{-strongly.} \quad (55)$$

Now fix a function  $\varphi \in C_0^\infty(\mathbb{R}^d)$  with  $\varphi = 1$  on  $B(R)$ . Then by (12)

$$\begin{aligned} \int_{\mathbb{R}^d} |u_n - u|^2 \mu_R(dx) &= \int_{\mathbb{R}^d} |u_n \varphi - u \varphi|^2 \mu_R(dx) \\ &\leq \epsilon \mathcal{E}^{(\alpha)}(u_n \varphi - u \varphi, u_n \varphi - u \varphi) + M(\epsilon) \int_{\mathbb{R}^d} |u_n \varphi - u \varphi|^2 dx. \end{aligned}$$

The second term of the right hand side converges to 0 as  $n \rightarrow \infty$  by (55), and Lemma 8.8 prove

$$\sup_n \mathcal{E}^{(\alpha)}(u_n \varphi - u \varphi, u_n \varphi - u \varphi) < \infty.$$

Hence the sequence  $\{u_n\}$  is  $L^2(\mu_R)$ -convergent to  $u$  because  $\epsilon$  is arbitrary.

Moreover, since by Theorem 3.5,

$$\begin{aligned} \int_{\mathbb{R}^d} |u_n - u|^2 \mu(dx) &= \int_{\mathbb{R}^d} |u_n - u|^2 \mu_R(dx) + \int_{\mathbb{R}^d} |u_n - u|^2 \mu_{R^c}(dx) \\ &\leq \int_{\mathbb{R}^d} |u_n - u|^2 \mu_R(dx) + \|G \mu_{R^c}\|_\infty \mathcal{E}^{(\alpha)}(u_n - u, u_n - u), \end{aligned}$$

we have

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^d} |u_n - u|^2 \mu(dx) \leq \limsup_{n \rightarrow \infty} \|G \mu_{R^c}\|_\infty \mathcal{E}^{(\alpha)}(u_n - u, u_n - u).$$

By the definition of  $\mathcal{K}_d^\infty$  the right hand side converges to 0 as  $R \rightarrow \infty$ , which proves that  $\{u_n\}$  is an  $L^2(\mu)$ -convergent sequence to  $u$ .  $\square$

To prove the continuity of  $h$ , we use a theorem of Bass and Levin [4].

**Theorem 8.11** (cf. [4]). *If  $h$  is a bounded  $(-\Delta)^{\frac{\alpha}{2}}$ -harmonic function on a ball  $B(x_0, 2)$ , then  $h$  is Hölder continuous in  $B(x_0, 1)$ ; there exists  $c_1$  and  $\beta > 0$  such that*

$$|h(x) - h(y)| \leq c_1 \|h\|_\infty |x - y|^\beta, \quad x, y \in B(x_0, 1).$$

By lemma 5.14, since  $E_x[h(X_{\tau_D})]$  is  $(-\Delta)^{\frac{\alpha}{2}}$ -harmonic on  $D$ , it is continuous on  $D$  by Theorem 8.11, thus it is enough to prove the continuity of  $E_x[\int_0^{\tau_D} h(X_t)dA_t^\mu]$ . We put  $f(x) = E_x[\int_0^{\tau_D} h(X_t)dA_t^\mu]$ . Although  $f(x)$  is continuous by [5, Proposition 6.6], we give an alternative proof of the continuity of  $f(x)$ .

Set

$$k^D(x, z) = E_x[G(z, X_{\tau_D})].$$

**Lemma 8.12.** *Let  $K$  be a compact subset of  $D$ . Then it follows that*

$$\sup_{x \in \mathbb{R}^d, z \in K} k^D(x, z) \leq \frac{c_d}{\text{dist}(K, D^c)^{d-\alpha}}.$$

*Proof.* For  $z \in K$  and  $X_{\tau_D} \in D^c$

$$\text{dist}(K, D^c) \leq |z - X_{\tau_D}|.$$

Hence the lemma follows from  $G(z, y) = \frac{c(d)}{|z-y|^{d-\alpha}}$ .  $\square$

Let  $k_z^D(x)$ ,  $z \in K$ , denotes  $k^D(x, z)$ . It is clear that  $k_z^D(x)$  is harmonic on  $D$  and bounded on  $\mathbb{R}^d$ . Thus by theorem 8.11, we have the next lemma.

**Lemma 8.13.** *Let  $D'$  be a domain with  $K \subset D' \subset \bar{D}' \subset D$ . For all  $x, y \in D'$ ,  $z \in K$ , there exist  $c_1 > 0$  and  $\beta > 0$  such that*

$$|k_z^D(x) - k_z^D(y)| \leq c_1 \sup_{x \in \mathbb{R}^d} |k_z^D(x)| |x - y|^\beta.$$

Let us define a Kernel  $G_l^D$ ,  $l > 0$ , by

$$G_l^D(x, z) = G(x, z) \wedge (c_d l) - k_z^D(x).$$

**Lemma 8.14.** *Let  $K$  and  $D'$  be the sets in Lemma 8.13. For any  $z \in K$ ,  $x, y \in D'$*

$$|G_l^D(x, z) - G_l^D(y, z)| \leq c(l, K) |x - y|^\beta.$$

*Proof.* By the definition of  $G_l^D$ ,

$$\begin{aligned} & |G(x, z) \wedge (c_d l) - G(y, z) \wedge (c_d l)| \\ &= c_d \left| \frac{1}{|x - z|^{d-\alpha}} \wedge l - \frac{1}{|y - z|^{d-\alpha}} \wedge l \right| \\ &= c_d \left| \frac{1}{|x - z|^{d-\alpha} \vee \frac{1}{l}} - \frac{1}{|y - z|^{d-\alpha} \vee \frac{1}{l}} \right| \\ &= c_d \left| \frac{|y - z|^{d-\alpha} \vee \frac{1}{l} - |x - z|^{d-\alpha} \vee \frac{1}{l}}{(|x - z|^{d-\alpha} \vee \frac{1}{l}) (|y - z|^{d-\alpha} \vee \frac{1}{l})} \right| \\ &\leq l^2 c_d \left| |y - z|^{d-\alpha} \vee \frac{1}{l} - |x - z|^{d-\alpha} \vee \frac{1}{l} \right| \end{aligned}$$

Let  $F(t) = t^{d-\alpha} \vee \frac{1}{l}$ ,  $t \geq 0$ . Then since

$$|F(a) - F(b)| \leq \sup_{t \in [a, b]} |F'(t)| |a - b|,$$

the last term is less than or equal to

$$c(l) ||y - z| - |x - z|| \leq C(l) |y - x|.$$

Hence lemma 8.13 leads us to the lemma. □

**Lemma 8.15.** *The function  $G^D(h\mu)(x)$  can be approximated by  $G_l^D(h\mu)(x)$ ;*

$$\limsup_{l \rightarrow \infty} \sup_{x \in D} |G^D(h\mu)(x) - G_l^D(h\mu)(x)| = 0.$$

*Proof.* By Dynkin's formula, we have

$$\begin{aligned} & |G^D(h\mu)(x) - G_l^D(h\mu)(x)| \\ &= \left| \int_D G(x, z) h(z) \mu(dz) - \int_D (G(x, z) \wedge c_d l) h(z) \mu(dz) \right|. \end{aligned}$$

Let  $\alpha_l = (\frac{1}{l})^{1/(d-\alpha)}$ . Then the right hand side is equal to

$$\begin{aligned} & \left| \int_{\{|x-z| \leq \alpha_l\} \cap D} G(x, z) h(z) \mu(dz) - \int_{\{|x-z| \leq \alpha_l\} \cap D} (c_d l) h(z) \mu(dz) \right| \\ & \leq 2 \int_{\{|x-z| \leq \alpha_l\} \cap D} G(x, z) h(z) \mu(dz) \\ & \leq 2 \|h\|_\infty \sup_{x \in D} \int_{\{|x-z| \leq \alpha_l\} \cap D} G(x, z) \mu(dz). \end{aligned}$$

Since  $\mu \in \mathcal{K}_d$ ,

$$\limsup_{l \rightarrow \infty} \sup_{x \in D} \int_{\{|x-z| \leq \alpha_l\} \cap D} G(x, z) \mu(dz) = 0.$$

Therefore the proof is completed. □

**Lemma 8.16.** *For any compact set  $K \subset D$*

$$|G_l^D(h\mu)(x) - G_l^D(hI_K\mu)(x)| \leq c_d l \|h\|_\infty \mu(D \setminus K).$$

*Proof.* By the definition of  $G_l^D$ , we have

$$\begin{aligned} |G_l^D(h\mu)(x) - G_l^D(hI_K\mu)(x)| & \leq \int_{D \setminus K} G_l^D(x, z) h(z) \mu(dz) \\ & \leq c_d l \|h\|_\infty \mu(D \setminus K). \end{aligned}$$

□

**Lemma 8.17.** *The function  $G_l^D(hI_K\mu)(x)$  is Hölder continuous with order  $\beta$ ;*

$$|G_l^D(hI_K\mu)(x) - G_l^D(hI_K\mu)(y)| \leq c(l, K)\|h\|_\infty|x - y|^\beta, \quad x, y \in D',$$

where  $\beta$  is the constant which appears in Theorem 8.11.

*Proof.* Using lemma 8.14, we have

$$\begin{aligned} |G_l^D(hI_K\mu)(x) - G_l^D(hI_K\mu)(y)| &\leq \int |G_l^D(x, z) - G_l^D(y, z)|h(z)I_K(z)\mu(dz) \\ &\leq c_1(l, K)\|h\|_\infty|x - y|^\beta\mu(K) \end{aligned}$$

Putting  $c(l, K) = c_1(l, K)\mu(K)$ , we obtain this lemma.  $\square$

**Proposition 8.18.** *The function  $h$  is continuous.*

*Proof.* Let  $D$  be relatively compact domain. Since  $E_x[h(X_{\tau_D})]$  is  $(-\Delta)^{\frac{\alpha}{2}}$ -harmonic on  $D$ , it is continuous on  $D$  by theorem 8.11. On account of Lemma 5.14, we have only to deal with  $G^D(h\mu)(x) (= E_x[\int_0^{\tau_D} h(X_t)dA_t^\mu])$ .

First note

$$\begin{aligned} &|G^D(h\mu)(x) - G^D(h\mu)(y)| \\ &\leq |G^D(h\mu)(x) - G_l^D(h\mu)(x)| \\ &\quad + |G_l^D(h\mu)(x) - G_l^D(hI_K\mu)(x)| + |G_l^D(hI_K\mu)(x) - G_l^D(hI_K\mu)(y)| \\ &\quad + |G_l^D(hI_K\mu)(y) - G_l^D(h\mu)(y)| + |G_l^D(h\mu)(y) - G^D(h\mu)(y)|. \end{aligned}$$

By lemma 8.15, for any  $\epsilon > 0$  there exists  $l_0 \in \mathbb{N}$  such that

$$l \geq l_0 \implies \sup_{x \in D} |G^D(h\mu)(x) - G_l^D(h\mu)(x)| \leq \frac{\epsilon}{5} \quad (56)$$

Next, let  $\{K_r\}_{r>0}$  be a increasing sequence of relatively compact open subset of  $D$  such that  $K_r \uparrow D$  as  $r \rightarrow \infty$ . By lemma 8.16, for any  $\epsilon > 0$  there exists a  $r_0 \in \mathbb{N}$  such that

$$r \geq r_0 \implies \sup_{x \in D} |G_l^D(h\mu)(x) - G_l^D(hI_{K_r}\mu)(x)| < \frac{\epsilon}{5}. \quad (57)$$

Finally, by Lemma 8.17, for any  $\epsilon > 0$  there exists  $\delta > 0$  such that for all  $x, y \in D'$

$$|x - y| \leq \delta \implies |G_l^D(hI_K\mu)(x) - G_l^D(hI_K\mu)(y)| \leq \frac{\epsilon}{5}. \quad (58)$$

Combining (56), (57) and (58), we see that for any  $\epsilon > 0$  there exists  $\delta > 0$  such that for all  $x, y \in D'$  with  $|x - y| \leq \delta$ ,

$$|G^D(h\mu)(x) - G^D(h\mu)(y)| \leq \epsilon.$$

Since  $D$  and  $D'$  are arbitrary, the function  $h$  is continuous on  $\mathbb{R}^d$ .  $\square$



Now we consider asymptotic of  $h$  as  $|x| \rightarrow \infty$ . Let  $w$  be a positive continuous function with compact support. Suppose that  $0 \in \text{supp}[w] \subset B(R)$ . By Theorem 5.11 and the continuity of  $h$

$$c \int_{B(R)} G^\nu(x, y)w(y)dy \leq h(x) \leq C \int_{B(R)} G^\nu(x, y)w(y)dy,$$

and so by the inequality (24),

$$c \int_{B(R)} G(x, y)w(y)dy \leq h(x) \leq C \int_{B(R)} G(x, y)w(y)dy.$$

The Harnack inequality to  $\{G(x, \cdot)\}_{\{x \in B(R)^c\}}$  says that for any  $x \in B(R)^c$  and  $y \in \text{supp}[w]$

$$cG(x, y) \leq G(x, 0) \leq CG(x, y).$$

Therefore we see that

$$cG(x, 0) \leq h(x) \leq CG(x, 0) \quad \text{for } x \in B(R)^c,$$

namely,

$$\frac{c}{|x|^{d-\alpha}} \leq h(x) \leq \frac{C}{|x|^{d-\alpha}} \quad \text{for } x \in B(R)^c. \quad (59)$$

The equation (59) implies the following theorem.

**Theorem 8.19.** *The operator  $\mathcal{H}^{\lambda+\mu}$  is null critical if and only if  $\alpha < d \leq 2\alpha$ .*

Therefore if  $\alpha < d \leq 2\alpha$  and  $\mu = \mu^+ - \mu^- \in \mathcal{K}_d^\infty - \mathcal{K}_d^\infty$ , we obtain the differentiability of spectral functions.

Now we introduce a concrete example.

**Example 8.20** ([50, Example 5.5], [20]). *Let  $d = 1$  and  $\alpha > 1$ . Then  $\mathbf{M}$  is recurrent and  $\lambda^+ = 0$ . When  $\mu = -\delta_0$ , the Dirac measure at the origin,  $\mu$  belongs to  $\mathcal{K}_1^\infty$  and the corresponding additive functional is identical to the local time at the origin. For  $\lambda > 0$ , the principal eigenvalue of  $\frac{1}{2}(-\Delta)^{\alpha/2} - \lambda\delta_0$  is calculated in [40]:*

$$C(\lambda) = \begin{cases} \left( \frac{2^{1/\alpha}}{\alpha \sin(\frac{\pi}{\alpha})} \right)^{\frac{\alpha}{\alpha-1}} \lambda^{\frac{\alpha}{\alpha-1}} & \lambda > 0 \\ 0 & \lambda \leq 0. \end{cases}$$

So the Legendre transform of  $C(\lambda)$  is

$$I(\theta) = \begin{cases} \frac{(\alpha-1)^{(\alpha-1)}}{2} \left( \sin \frac{\pi}{\alpha} \right)^\alpha \theta^\alpha & x > 0 \\ 0 & x \leq 0. \end{cases}$$

### 8.3 Relativistic $\alpha$ -stable processes

In this subsection, we refer to the paper [45]. If  $\mathbf{M}$  has a symbol  $(|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m$  ( $m > 0$ ), the symmetric Markov process  $\mathbf{M}$  is called a *relativistic  $\alpha$ -stable process*. The generator of this process is denoted by  $\mathcal{H} = m - (-\Delta + m^{2/\alpha})^{\alpha/2}$ . Since it is non-local, this process is a pure jump process. We denote by  $\mathbf{M}^{(\alpha),r}$ . Let  $(\mathcal{R}^{(\alpha)}, \mathcal{D}(\mathcal{R}^{(\alpha)}))$  be the Dirichlet form associated with  $\mathbf{M}^{(\alpha),r}$ . By [19, Example 1.4.1], it is denoted by

$$\begin{aligned} \mathcal{R}^{(\alpha)}(u, v) &= \int_{\mathbb{R}^d} |\hat{u}(\xi)| |\overline{\hat{v}(\xi)}| ((|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m) d\xi \\ \mathcal{D}(\mathcal{R}^{(\alpha)}) &= \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |u(\hat{\xi})|^2 ((|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m) d\xi < \infty \right\}, \end{aligned}$$

where  $\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{ix \cdot \xi} u(x) dx$  is the Fourier transform of  $u$ . Since the symbol  $p(\xi)$  satisfies

$$p(\xi) \sim c(d, \alpha, m) |\xi|^2 \text{ as } |\xi| \rightarrow 0 \quad (c(d, \alpha, m) \text{ is constant}),$$

where  $a(x) \sim b(x)$   $x \rightarrow \infty$  (resp.  $x \rightarrow 0$ ) means that

$$\lim_{x \rightarrow \infty} \frac{a(x)}{b(x)} = 1, \quad \left( \text{resp. } \lim_{x \rightarrow 0} \frac{a(x)}{b(x)} = 1 \right).$$

Then we find that  $\mathbf{M}^{(\alpha),r}$  is transient if and only if  $d > 2$  by [19]. So in the case of  $\mathbf{M}^{(\alpha),r}$ , transience and recurrence are equivalent to the case of Brownian motion.

In Ryznar [36], they decided the Lévy measure for  $\mathbf{M}^{(\alpha),r}$  so by [19, Example 1.4.1], we know that

$$\mathcal{R}^{(\alpha)}(u, v) = \frac{1}{2} c(d, \alpha) \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} \psi(m^{1/\alpha} |x - y|) dx dy$$

where

$$\psi(r) = \frac{I(r)}{I(0)}, \quad I(r) = \int_0^\infty s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4} - \frac{r^2}{s}} ds. \quad (60)$$

Let  $\phi(\lambda)$  be the Laplace exponent of the relativistic  $\alpha$ -stable subordinator  $T_t$ , that is,  $\phi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$  and

$$E[\exp(-\lambda T_t)] = \exp(-t\phi(\lambda)).$$

Since  $\phi(\lambda)$  is a complete Bernstein function,  $\phi(\lambda)$  is represented by

$$\phi(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t}) \mu(dt), \quad \forall \lambda > 0,$$

where  $\mu$  is a  $\sigma$ -finite measure on  $(0, \infty)$  satisfying

$$\int_0^\infty (t \wedge 1) \mu(dt) < \infty,$$

[cf. [21]]. It is clear that if  $\phi(\lambda)$  is relativistic  $\alpha$ -stable subordinator,

$$a = \lim_{\lambda \rightarrow 0} \phi(\lambda) = 0, \quad b = \lim_{\lambda \rightarrow \infty} \frac{\phi(\lambda)}{\lambda} = 0.$$

Since  $b = 0$  and  $\lim_{\lambda \rightarrow \infty} \phi(\lambda) = \infty$ , we must have  $\mu((0, \infty)) = \infty$ , and  $\phi(\lambda)$  satisfies

$$\begin{aligned} \phi(\lambda) &\sim c_1 \lambda, & \lambda \rightarrow 0 \\ \phi(\lambda) &\sim c_2 \lambda^{\alpha/2}, & \lambda \rightarrow \infty. \end{aligned}$$

So using Theorem 3.1 and Theorem 3.3 in [32], we know that the asymptotic behavior of  $G(x, y)$  as follows;

**Lemma 8.21.**

$$G(x, y) \sim C_1(d, \alpha) |x - y|^{\alpha-d}, \quad |x - y| \rightarrow 0 \quad (61)$$

$$G(x, y) \sim C_2(d, \alpha) |x - y|^{2-d}, \quad |x - y| \rightarrow \infty. \quad (62)$$

Note that the Kato class  $\mathcal{K}_d$  associated with  $\mathbf{M}^{(\alpha),r}$  is equivalent to the one associated with the standard stable processes by (61).

Now we concern with the property of Kato class.

**Lemma 8.22.** *Let  $G(x, y) = G(|x - y|)$  be the Green function of the process  $\mathbf{M}^{(\alpha),r}$ . Then there exists a constant  $b > 1$  such that*

$$\limsup_{r \rightarrow 0} \frac{G(br)}{G(r)} < 1.$$

*Proof.* Let  $b > 1$ . Since  $G(r) \sim c(d)r^{\alpha-d}$  as  $r \rightarrow 0$ ,

$$\frac{G(br)}{G(r)} = \frac{G(br)}{c(d)(br)^{\alpha-d}} \cdot \frac{c(d)r^{\alpha-d}}{G(r)} \cdot b^{\alpha-d} \sim b^{\alpha-d}.$$

Hence

$$\limsup_{r \rightarrow 0} \frac{G(br)}{G(r)} = b^{\alpha-d} < 1.$$

□

By Lemma 8.22, we can confirm that Proposition 3.3 holds in this case.

Now we prove the continuity of  $h$ .

**Proposition 8.23.** *For  $\mu \in \mathcal{K}_d$ ,  $P_t^\mu$  is strong Feller semigroup, that is,  $P_t^\mu$  maps  $\mathcal{B}_b$  to  $C_b$ .*

*Proof.* We follow the method of the Chung and Zhao [14].

Since the semigroup  $P_t^\mu$  maps  $\mathcal{B}_b$  to  $\mathcal{B}_b$  on account of [2, Theorem 6.1] and the unperturbed semigroup  $P_t$  has strong Feller property, it is sufficient to prove that for  $f \in \mathcal{B}_b$  and  $0 < \delta < t$ ,  $P_\delta(P_{t-\delta}^\mu f)(x)$  converges uniformly to  $P_t^\mu f(x)$  as  $\delta \rightarrow 0$ .

By proposition 3.3, if  $\mu = \mu^+ - \mu^- \in \mathcal{K}_d - \mathcal{K}_d$ , then for sufficiently small  $t > 0$ , we can take

$$\sup_{x \in \mathbb{R}^d} E_x[A_t^{|\mu|}] := \alpha < 1.$$

K'hasminkii's lemma tells us that

$$1 \leq \sup_{x \in \mathbb{R}^d} E_x[\exp(A_t^{|\mu|})] \leq \frac{1}{1 - \alpha}.$$

Using the proposition 3.3 again, we find that

$$\limsup_{t \rightarrow 0} \sup_{x \in \mathbb{R}^d} E_x[\exp(A_t^{|\mu|})] = 1. \quad (63)$$

By the Markov property, for  $0 < \delta < t$

$$\begin{aligned} P_\delta P_{t-\delta}^\mu f(x) &= E_x [E_{X_\delta} [\exp(-A_{t-\delta}^\mu) f(X_{t-\delta})]] \\ &= E_x [\exp(-A_t^\mu + A_\delta^\mu) f(X_t)] \end{aligned}$$

Hence we have by Cauchy-Schwarz inequality,

$$\begin{aligned} |P_t^\mu f(x) - P_\delta P_{t-\delta}^\mu f(x)| &\leq E_x \left[ \exp(-A_t^\mu) |\exp(A_\delta^{|\mu|}) - 1| \right] \|f\|_\infty \\ &\leq E_x [\exp(-2A_t^\mu)]^{1/2} E_x [\exp(2A_t^{|\mu|}) - 1]^{1/2} \end{aligned}$$

This converges to zero uniformly in all  $x$  by (63). We obtain this lemma.  $\square$

From the argument in §5, we can construct a finely continuous bounded ground state  $h$ . By Proposition 8.23, we obtain the following proposition.

**Proposition 8.24.** *The function  $h$  is continuous.*

*Proof.* Since  $P_t^\mu$  does not possess the Green function,  $h$ -transformed semigroup  $P_t^{\mu,h}$  does not possess one, too. On account of the excessiveness of  $h$ , we know that  $P_t^{\mu,h}$  generates the Markov process. The Markov process is recurrent because of no existence of Green function. So it is conservative. For any  $t > 0$ , we know that

$$P_t^{\mu,h} 1 = 1 \quad \iff \quad \frac{1}{h} P_t^\mu h = 1.$$

Hence we obtain

$$P_t^\mu h = h.$$

Since  $h$  is bounded and  $\mu \in \mathcal{K}_{d,\alpha}$ , we know that  $h$  is continuous by Proposition 8.23.  $\square$

Now we state the asymptotic behavior of the function  $h$  at infinity. For a non-negative bounded Borel function  $w \not\equiv 0$  with compact support, define  $\nu = \mu + wdx$ . Then we get the following equality similarly;

$$h(x) = \int_{\mathbb{R}^d} G^\nu(x, y)h(y)w(y)dy. \quad (64)$$

**Lemma 8.25.** *There are positive constants  $C_1$ ,  $C_2$  and  $R > 0$  such that*

$$\frac{C_1}{|x|^{d-2}} \leq h(x) \leq \frac{C_2}{|x|^{d-2}}, \quad |x| > R.$$

*Proof.* Let  $w \not\equiv 0$  be a positive continuous function with compact support. Suppose that  $0 \in \text{supp}[w] \supset B(R)$ . Then by (64) and the continuity of  $h$ ,

$$c \int_{B(R)} G^\nu(x, y)w(y)dy \leq h(x) \leq C \int_{B(R)} G^\nu(x, y)w(y)dy.$$

Since we know that

$$G^\nu(x, y) \sim G(x, y), \quad \text{for } x \neq y$$

on account of [47], we have

$$c \int_{B(R)} G(x, y)w(y)dy \leq h(x) \leq C \int_{B(R)} G(x, y)w(y)dy.$$

Since  $\{G(x, \cdot)\}_{x \in B(R)^c}$  is harmonic on  $\text{supp}[w]$ , the Harnack inequality [43] to  $\{G(x, \cdot)\}_{x \in B(R)^c}$  says that for any  $x \in B(R)^c$  and  $y \in \text{supp}[w]$

$$cG(x, y) \leq G(x, 0) \leq CG(x, y).$$

Therefore we see that

$$C_1G(x, 0) \leq h(x) \leq C_2G(x, 0), \quad \text{for } x \in B(R)^c$$

namely,

$$\frac{C_1}{|x|^{d-2}} \leq h(x) \leq \frac{C_2}{|x|^{d-2}}, \quad \text{for } x \in B(R)^c.$$

□

**Theorem 8.26.** *If  $d = 3, 4$  and  $\mu = \mu^+ - \mu^- \in \mathcal{S}_\infty - \mathcal{S}_\infty$ , the Schrödinger type operator  $\mathcal{H}^{\lambda+\mu}$  is null critical.*

Therefore if  $d = 3, 4$ , we obtain the differentiability of spectral functions.

Now we check the relation between the spectral function and the logarithmic moment generating function.

**Lemma 8.27.** *Let  $J(dx)$  be the Lévy measure of  $\mathbf{M}^{(\alpha),r}$ . Then  $J(dx)$  is exponentially localized, that is there exists a positive constant  $\delta$  such that*

$$\int_{|x|>1} e^{\delta|x|} J(dx) < \infty$$

*Proof.* By [36] or [12], we know that

$$J(dx) = \frac{c(d, \alpha)}{|x|^{d+\alpha}} \psi(m^{1/\alpha}|x|) dx,$$

where  $\psi(r)$  is defined in (60). Since  $\psi(m^{1/\alpha}|x|) \sim e^{-m^{1/\alpha}|x|} (1 + (m^{1/\alpha}|x|)^{\frac{d+\alpha-1}{2}})$  as  $|x| \rightarrow \infty$  (cf. [12]), taking  $\delta < m^{1/\alpha}$ , we have

$$\begin{aligned} \int_{|x|>1} e^{\delta|x|} J(dx) &= c(d, \alpha) \int_{|x|>1} \frac{e^{(\delta-m^{1/\alpha})|x|} \psi(m^{1/\alpha}|x|)}{|x|^{d+\alpha}} dx \\ &\leq C \int_1^\infty e^{(\delta-m^{1/\alpha})r} (1 + m^{1/\alpha}r)^{\frac{d+\alpha-1}{2}} r^{-1-\alpha} dr < \infty. \end{aligned}$$

The proof is completed. □

Using the argument in [46], we have the following proposition.

**Proposition 8.28.** *Let  $\mu$  be a signed measure which both positive and negative parts belong Kato class  $K_{d,\alpha}$ . Then it holds that*

$$C(\lambda) = \lim_{t \rightarrow \infty} \frac{1}{t} E_x[\exp(-\lambda A_t^\mu)].$$

Using the Gärtner-Ellis theorem [17, Theorem 2.3.6] and Theorem 2.1, we obtain the Corollary 2.2.

Finally, we introduce a way to construct an example.

From now on, we follow the method of Shiozawa [40]. Suppose that  $d = 1$  and  $1 < \alpha \leq 2$ . Let  $\mathcal{H}^{\lambda\delta_0} = (-\Delta + m^{2/\alpha})^{\alpha/2} - m - \lambda\delta_0$ , where  $\delta_0$  is the Dirac measure at 0.

We state the method to calculate the spectral function (or the principal eigenvalue of  $-\mathcal{H}^{\lambda\delta_0}$ )  $C(\lambda)$  for  $\lambda > 0$ . Let  $G_\beta$  be the  $\beta$ -resolvent of  $\mathbf{M}^{(\alpha),r}$ ,

$$G_\beta(x, y) = \frac{1}{\pi} \int_0^\infty \frac{\cos\{(x-y)z\}}{\beta + \{(z^2 + m^{2/\alpha})^{\alpha/2} - m\}} dz.$$

Moreover, in generally, the resolvent  $G_{C(\lambda)}$  satisfies

$$h(x) = \lambda \int_{\mathbb{R}} G_{C(\lambda)}(x, y) h(y) \mu(dy),$$

where  $h$  is the ground state of  $C(\lambda)$ . So when  $\lambda\mu$  is  $\lambda\delta_0$ , we know that

$$h(x) = \lambda G_{C(\lambda)}(x, 0) h(0).$$

Hence

$$\lambda G_{C(\lambda)}(0, 0) = 1.$$

Thus we find that the spectral function  $C(\lambda)$  is the solution of the following equation:

$$(G_{C(\lambda)}(0, 0) = ) \frac{1}{\lambda} = \frac{1}{\pi} \int_0^\infty \frac{1}{C(\lambda) + \{(z^2 + m^{2/\alpha})^{\alpha/2} - m\}} dz.$$

If we are able to solve the equation, we obtain the rate function  $I(\theta)$  by Legendre transform. Remark that for  $\lambda \leq 0$ ,  $C(\lambda)$  becomes 0 (cf. [48]). Since the Dirac measure (at 0)  $\delta_0$  corresponds to the local time (at 0)  $l_0(t)$ , we can establish the large deviation principle for the local time  $l_0(t)$ .

## 9 Remark

In this section, we would like to point out that the differentiability of spectral function is just sufficient condition not necessary condition to hold LDP.

We introduce a counter-example. When  $\mathbf{M}$  is the  $d$ -dimensional Brownian motion  $d \geq 5$  and  $\mu = \mu^+ - \mu^- \in \mathcal{K}_d^\infty - \mathcal{K}_d^\infty$ , the Schrödinger type operator  $\mathcal{H}^{\lambda^\pm}$  are positive critical. Hence the spectral function  $C(\lambda)$  is not differentiable. But in [49, Theorem 2.4, Theorem 2.5], Takeda prove the LDP for  $A_t^\mu$  in wider class with respect to  $\mu$ .

**Theorem 9.1** (Theorem 2.4, Theorem 2.5 [49]). *Let  $I(\theta)$  be the Legendre transform of  $C(\lambda)$ , that is,*

$$I(\theta) = \sup_{\lambda \in \mathbb{R}} \{\theta\lambda - C(\lambda)\}.$$

(I) *Let  $\mu = \mu^+ - \mu^- \in \mathcal{K}_d - \mathcal{K}_{d,loc}$ . Then for any open set  $G$*

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log P_x \left( \frac{A_t^\mu}{t} \in G \right) \geq - \inf_{\theta \in G} I(\theta).$$

(II) *Let  $\mu = \mu^+ - \mu^- \in \mathcal{K}_d - \mathcal{K}_d$ . Then for any closed set  $F$*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log P_x \left( \frac{A_t^\mu}{t} \in F \right) \leq - \inf_{\theta \in F} I(\theta).$$

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