CHOQUET INTEGRATION WITH SUBMODULAR FUNCTION ON MEASURABLE SPACE WITH SIGMA-ALGEBRA GENERATING CHAIN

Tetsuya Hattori

Abstract

Based on a study of a formula representing submodular set function as a supremum of measures dominated by the set function, we present a corresponding formula for a Choquet integration with respect to the set function, on a measurable space which has a chain of measurable set generating the sigma-algebra. As an application we reproduce a basic formula in mathematical finance on law invariant coherent risk measures. We also study a recursion relation of set functions for which the representation formula characterizes the fixed point.

1 Introduction

Let $(\Omega, \mathcal{F})$ be a measurable space, namely, a σ-algebra $\mathcal{F}$ is a class of subsets of Ω and is closed under complements and countable unions. For a measurable set $A \in \mathcal{F}$ denote by $\mathcal{F}|_A$, the class of measurable sets restricted to A, and denote the set of finite measures on the measurable space $(A, \mathcal{F}|_A)$ by $\mathcal{M}(A)$.

For a real valued set function $v : \mathcal{F} \to \mathbb{R}$ and a measurable set $A \in \mathcal{F}$, let $\mathcal{C}_{-,v}(A)$ be a class of measures dominated by v on A;

$$\mathcal{C}_{-,v}(A) = \{\mu \in \mathcal{M}(A) \mid \mu(A) = v(A),\ \mu(B) \leqq v(B),\ B \in \mathcal{F}|_A\}. \tag{1}$$

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If

$$v(B) = \sup_{\mu \in \mathcal{C}_{-,v}(A)} \mu(B), \tag{2}$$

holds for all $A, B \in \mathcal{F}$ satisfying $B \subset A$, then it is easy to see that

$$v(A) + v(B) \geqq v(A \cup B) + v(A \cap B), \quad A, B \in \mathcal{F}, \tag{3}$$

holds [5, Proposition 1]. A set function which satisfies (3) is called a submodular function.

In [5, Theorem 3] we gave a proof of a converse that (3) implies (2), for a non-decreasing and continuous set function v, when $\mathcal{F}$ is generated by a chain. Here, by non-decreasing we mean

$$v(A) \leqq v(B), \ A \subset B, \ A, B \in \mathcal{F}. \tag{4}$$

We will later define the continuity of set functions which we adopt in this paper, and first make the assumption on $\mathcal{F}$ precise. We consider, as in [5], a following set of conditions for a class of measurable sets $\mathcal{I} \subset \mathcal{F}$;

$$\begin{cases} i) \ \ \sigma[\mathcal{I}] = \mathcal{F}, \text{ where } \sigma[\mathcal{I}] \text{ denotes the smallest } \sigma\text{-algebra containing } \mathcal{I}, \\ ii) \ \ \emptyset \in \mathcal{I}, \ \Omega \in \mathcal{I}, \\ iii) \ \ \mathcal{I} \text{ is a chain, i.e., totally ordered with respect to inclusion,} \\ \quad \text{i.e., for all } I_1, I_2 \in \mathcal{I} \text{ either } I_1 \subset I_2 \text{ or } I_2 \subset I_1 \, . \end{cases} \tag{5}$$

Let $\mathcal{X}$ denote the collection of such classes;

$$\begin{aligned} \mathcal{X} &= \{\mathcal{I} \subset \mathcal{F} \mid (5) \text{ holds}\} \\ &= \{\mathcal{I} \subset \mathcal{F} \mid \ \mathcal{I} \text{ is a chain such that } \ \emptyset \in \mathcal{I}, \ \Omega \in \mathcal{I}, \ \sigma[\mathcal{I}] = \mathcal{F}\}, \end{aligned} \tag{6}$$

and for $\mathcal{I} \in \mathcal{X}$ and $A \in \mathcal{F}$, define $\mathcal{I}_A$, the insertion of A into $\mathcal{I}$, by

$$\mathcal{I}_A = \{A \cap I \mid I \in \mathcal{I}\} \cup \{A \cup I \mid I \in \mathcal{I}\}. \tag{7}$$

Note that $\mathcal{X}$ is closed under the insertion; if $\mathcal{I} \in \mathcal{X}$ and $A \in \mathcal{F}$ then $A \in \mathcal{I}_A \in \mathcal{X}$ [5, Lemma 2]. With these notations, [5, Proposition 1] and [5, Theorem 3] imply that if $\mathcal{X} \neq \emptyset$, then a non-decreasing continuous set function v is submodular, if and only if (2) holds for all $A, B \in \mathcal{F}$ satisfying $B \subset A$. See [3, 4, 8] and the references in [5] for a part of large background and long history of the basic theory of submodular functions.

It is proved in [5, Proposition 7] that Polish spaces (separable, complete, metric space) with Borel σ-algebra (the smallest σ-algebra containing all open balls) are examples of measurable spaces satisfying $\mathcal{X} \neq \emptyset$, so that the above mentioned equivalence between submodular property (3) and the representation formula (2) hold for Polish spaces. One

dimensional Borel σ-algebra and a finite set $\Omega_m = \{1, 2, \ldots, m\}$ with m elements and $\mathcal{F} = 2^{\Omega_m}$ are among examples of Polish spaces, as well as a wide class of spaces extensively used in the theory of stochastic processes. In this paper we exclusively consider $(\Omega, \mathcal{F})$ satisfying $\mathcal{X} \neq \emptyset$, and advance our study in [5] in the following directions.

- The definition (1) of $\mathcal{C}_{-,v}(A)$ implies that (2) follows from (3) if there exists a measure $\mu \in \mathcal{C}_{-,v}(A)$ such that $v(B) = \mu(B)$ holds. Such a measure is studied by Shapley [8] as an extremal measure when Ω is a finite set. In our formulation, the extremal measure of v is defined for each $\mathcal{I} \in \mathcal{X}$. (In [5, Theorem 3], for the sake of conciseness of the statement of the theorem, the notion of extremal measures is not explicitly written, and is implicitly introduced in the proof of the theorem.) For a non-decreasing set function v, we define $\mu_{v,\mathcal{I}} : \mathcal{I} \to \mathbb{R}_+$ by

  $$\mu_{v,\mathcal{I}}(I) = v(I), \ I \in \mathcal{I}. \tag{8}$$

  To extend $\mu_{v,\mathcal{I}}$ to a measure, we define in Definition 2.1 in § 2, that a set function v is continuous, if for every $\mathcal{I} \in \mathcal{X}$ $\mu_{v,\mathcal{I}}$, defined by (8) on $\mathcal{I}$ and uniquely extended to the finite algebra $\mathcal{J}$ generated by $\mathcal{I}$ as a finitely additive measure, is continuous on $\mathcal{J}$ in the standard measure theory sense. (We will see in § 2 that this definition is a generalization of the definition of continuity given in [5] for submodular functions.) If v is continuous then $\mu_{v,\mathcal{I}}$ is uniquely extended to a measure on $\mathcal{F}$, which we call the extremal measure of v corresponding to $\mathcal{I} \in \mathcal{X}$. We prove in § 2 another representation formula for submodular function in terms of the extremal measures

  $$v = \sup_{\mathcal{I} \in \mathcal{X}} \mu_{v,\mathcal{I}} \tag{9}$$

  which generalizes a corresponding result of theory of cores for a finite set in [8] to measurable spaces satisfying $\mathcal{X} \neq \emptyset$.

- We can define Choquet integration $v(f)$ with respect to non-decreasing continuous set function v for an integrable function f. If, in addition, v is submodular, the functional ρ defined by $\rho(f) = v(-f)$ satisfies the definition of the coherent risk measure studied in the field of mathematical finance. We then obtain a representation formula for ρ, based on (9), which reproduces a basic formula for a coherent risk measure with Fatou property studied in mathematical finance [1, 2, 6]. See [7] for generalization and further development on Choquet integration with respect to submodular functions including its relation to coherent risk measure. In [5] we briefly announced an outline of how these results are formulated in our framework, for which we give precise statements and proofs in § 3. As a further example in § 4, we reproduce in our framework a formula studied in [6] for law-invariant coherent risk measures.

- Suggested by the representation formula (9) for submodular functions, we consider the recursion relation

$$v_{n+1} = \sup_{\mathcal{I}\in\mathcal{X}} \mu_{v_n,\mathcal{I}}\,,\ n = 0, 1, 2, \ldots, \tag{10}$$

with $v_0 : \mathcal{F} \to \mathbb{R}$ being a non-decrasing continuous set function. Note that (9) implies $v_1 \neq v_0$ if v_0 is not submodular and $v_1 = v_0$ if v_0 is submodular. (We prove this equivalence in Theorem 2.7 in § 2.)

For a finite set $(\Omega_m, 2^{\Omega_m})$ all the non-decreasing set functions are continuous in our definition. We show in § 5 that if $m = 3$ and v_0 is non-decreasing then v_1 is submodular, hence $v_1 = v_2 = \cdots$ in (10), while we show an example of v_0 for $m = 4$ such that $v_1 \neq v_2$.

Note that while we define and state results in terms of submodular functions, the corresponding results in this paper for supermodular functions (convex games) hold through a well-known correspondence

$$\tilde{v}(A) = v(\Omega) - v(A^c) + v(\emptyset),\ A \in \mathcal{F}, \tag{11}$$

which gives a non-decreasing continuous supermodular (resp., submodular) function $\tilde{v}$ from a non-decreasing continuous submodular (resp., supermodular) function v satisfying $\tilde{v}(\Omega) = v(\Omega)$ and $\tilde{v}(\emptyset) = v(\emptyset)$. We will choose definitions and statements in § 2 so that corresponding results for convex games also hold by the correspondence (11). In fact, classical theories by Shapley [8], as well as many works on cooperative game theories, are written in terms of convex games, for which we should use (11) when comparing with the results for sub-modular functions.

For notational simplicity we assume $v(\emptyset) = 0$ for any set function v throughout this paper. The formula in this paper can be generalized to the case $v(\emptyset) \neq 0$ by the replacements $v(A) \mapsto v(A) - v(\emptyset)$.

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2 Representation formula for submodular function

In this section, we fix a measurable space $(\Omega, \mathcal{F})$ and assume $\mathcal{X} \neq \emptyset$, where $\mathcal{X}$ is defined in (6) in § 1.

Let $v:\ \mathcal{F}\to\mathbb{R}$ be a non-decreasing set function. Let $\mathcal{I}\in\mathcal{X}$, and denote by $\mathcal{J}$ the finitely additive class generated by $\mathcal{I}$, the smallest class of sets such that $\mathcal{I}$ is a subset and closed under complement and union. Since $\mathcal{I}$ is totally ordered with respect to inclusion, we have an explicit representation

$$\begin{aligned}\mathcal{J}=\{\bigcup_{i=1}^{n}(C_i\cap D_i^c)\mid\ & C_1\supset D_1\supset C_2\supset\cdots\supset D_n\,,\\ & C_i,\ D_i\in\mathcal{I}\,,\ i=1,2,\dots,n,\quad n=1,2,3,\dots\}.\end{aligned}\tag{12}$$

Using the notation in the right-hand side of (12), we can define, as in an elementary textbook on measure theory (see [5]), a finitely additive set function $\mu_{v,\mathcal{I}}:\ \mathcal{J}\to\mathbb{R}$ as a unique extension of (8), by

$$\mu_{v,\mathcal{I}}(\bigcup_{i=1}^{n}(C_i\cap D_i^c))=\sum_{i=1}^{n}(v(C_i)-v(D_i)).\tag{13}$$

Definition 2.1. We say that a non-decreasing set function $v:\ \mathcal{F}\to\mathbb{R}$ is continuous, if for every $\mathcal{I}\in\mathcal{X}$ the finitely additive measure $\mu_{v,\mathcal{I}}:\ \mathcal{J}\to\mathbb{R}$ defined by (13) is σ-additive, or equivalently, continuous, on the finitely additive class $\mathcal{J}$.

A standard extension theorem of measures and $\sigma[\mathcal{I}]=\mathcal{F}$ in (6) then imply that $\mu_{v,\mathcal{I}}$ is uniquely extended to a measure on $\mathcal{F}$ for all $\mathcal{I}\in\mathcal{X}$ if v is continuous. We will use the same notation and call the measure $\mu_{v,\mathcal{I}}:\ \mathcal{F}\to\mathbb{R}$ the extremal measure of v corresponding to $\mathcal{I}\in\mathcal{X}$.

We note that this definition of continuity of v is consistent with the definition of continuity in [5] for submodular functions. Namely, the following holds.

Proposition 2.2. *Let $(\Omega,\mathcal{F})$ be a measurable space satisfying $\mathcal{X}\neq\emptyset$ and $v:\ \mathcal{F}\to\mathbb{R}$ be a non-decreasing submodular function satisfying $v(\emptyset)=0$, i.e., a set function satisfying $v(\emptyset)=0$, (4) and (3). Then v is continuous if and only if*

$$\lim_{n\to\infty}v(A_n)=v(\bigcup_{n\in\mathbb{N}}A_n),\quad A_1\subset A_2\subset\cdots\,,\quad A_n\in\mathcal{F},\ n=1,2,3,\dots,\tag{14}$$

and

$$\lim_{n\to\infty}v(A_n)=v(\bigcap_{n\in\mathbb{N}}A_n)\quad A_1\supset A_2\supset\cdots\,,\quad A_n\in\mathcal{F},\ n=1,2,3,\dots,\tag{15}$$

hold.

We note that our definition of continuity implies (14) and (15) without an assmption of submodularity.

PROPOSITION 2.3. *Let $(\Omega, \mathcal{F})$ be a measurable space satisfying $\mathcal{X} \neq \emptyset$ and $v : \mathcal{F} \to \mathbb{R}$ be a non-decreasing continuous set function satisfying $v(\emptyset) = 0$. Then (14) and (15) hold.*

To prove Proposition 2.2 and Proposition 2.3 (as well as for later use) we prepare the following Lemma 2.4 and Lemma 2.5.

LEMMA 2.4. *Let $(\Omega, \mathcal{F})$ be a measurable space satisfying $\mathcal{X} \neq \emptyset$, and let $\mathcal{I} \in \mathcal{X}$. Denote by $\mathcal{J}$ the finitely additive class (12) generated by $\mathcal{I}$.*

If a non-decreasing set function $v : \mathcal{F} \to \mathbb{R}$ satisfies (3), i.e., submodular, then the finitely additive measure $\mu_{v,\mathcal{I}} : \mathcal{J} \to \mathbb{R}$ definedd by (13) satisfies $\mu_{v,\mathcal{I}}(J) \leqq v(J)$, $J \in \mathcal{J}$.

PROOF. A proof is basically same as that of [5, Lemma 6]. Using the expression (12) let

$$J = \bigcup_{i=1}^{n} (C_i \cap (D_i)^c) \in \mathcal{J}; \quad C_1 \supset D_1 \supset C_2 \supset \cdots \supset D_n\,,$$
$$C_i,\ D_i \in \mathcal{I}\,,\ i = 1, 2, \ldots, n,$$

and put $A_i = \bigcup_{j=i}^{n} (C_j \cap (D_j)^c)$, $i = 1, 2, \ldots, n$, and $A_{n+1} = \emptyset$. Then $A_1 = J$ and

$$A_i \cup D_i = C_i\,,\ A_i \cap D_i = A_{i+1}\,,\quad i = 1, 2, \ldots, n,$$

Apply (3) with $A = A_i$ and $B = D_i$ to find

$$v(C_i) + v(A_{i+1}) \leqq v(A_i) + v(D_i),\ i = 1, 2, \ldots, n.$$

Summing this up with i and using (13) leads to $\mu_{v,\mathcal{I}}(J) \leqq v(J)$, □

Let $(\Omega, \mathcal{F})$ be a measurable space satisfying $\mathcal{X} \neq \emptyset$, and let $\mathcal{I} \in \mathcal{X}$.

Before moving on to the next Lemma, we remark on sequential insertion to $\mathcal{I}$. For $A, B \in \mathcal{F}$, a sequential insertion of A and B to $\mathcal{I}$ in general depends on the order of the insertion. In fact,

$$\begin{aligned}(\mathcal{I}_A)_B &= \{I' \cap B \mid I' \in \mathcal{I}_A\} \cup \{I' \cup B \mid I' \in \mathcal{I}_A\} \\ &= \{I \cap (A \cap B) \mid I \in \mathcal{I}\} \cup \{(I \cap B) \cup (A \cap B) \mid I \in \mathcal{I}\} \\ &\quad \cup \{(I \cup B) \cap (A \cup B) \mid I \in \mathcal{I}\} \cup \{I \cup (A \cup B) \mid I \in \mathcal{I}\}\end{aligned} \tag{16}$$

implies $B \in (\mathcal{I}_A)_B$, while A may not be an element. Note, however, that (16) implies $A \cap B,\ A \cup B \in (\mathcal{I}_A)_B \cap (\mathcal{I}_B)_A$.

If $\{A, B\}$ is a chain, i.e., $A \subset B$ or $A \supset B$, then the order of insertion is irrelevant, and we will use the notation $\mathcal{I}_{A,B} := (\mathcal{I}_A)_B = (\mathcal{I}_B)_A$. For example, if $B \subset A$ then

$$\mathcal{I}_{A,B} := (\mathcal{I}_A)_B = \{I \cap B,\ (I \cup B) \cap A,\ I \cup A \mid I \in \mathcal{I}\} = (\mathcal{I}_B)_A \in \mathcal{X}. \tag{17}$$

For $\mathcal{A} = \{A_n \mid n = 1, 2, 3, \ldots\} \subset \mathcal{F}$ satisfying $A_1 \subset A_2 \subset \cdots$, define the (countable) insertion $\mathcal{I}_\mathcal{A}$ of $\mathcal{A}$ into $\mathcal{I}$ by

$$\mathcal{I}_\mathcal{A} = \{I \cap A_1 \mid I \in \mathcal{I}\} \cup \{(I \cup A_n) \cap A_{n+1} \mid I \in \mathcal{I},\ n \in \mathbb{N}\} \cup \{I \cup \bigcup_{n\in\mathbb{N}} A_n \mid I \in \mathcal{I}\}. \tag{18}$$

Similarly, For $\mathcal{A} = \{A_n \mid n = 1, 2, 3, \ldots\} \subset \mathcal{F}$ satisfying $A_1 \supset A_2 \supset \cdots$, define

$$\mathcal{I}_\mathcal{A} = \{I \cap \bigcap_{n\in\mathbb{N}} A_n \mid I \in \mathcal{I}\} \cup \{(I \cup A_{n+1}) \cap A_n \mid I \in \mathcal{I},\ n \in \mathbb{N}\} \cup \{I \cup A_1 \mid I \in \mathcal{I}\}. \tag{19}$$

LEMMA 2.5. *$\mathcal{I}_\mathcal{A}$ of (18) satisfies $A_n \in \mathcal{I}_\mathcal{A}$, $n \in \mathbb{N}$, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{I}_\mathcal{A}$, and $\mathcal{I}_\mathcal{A} \in \mathcal{X}$.*

$\mathcal{I}_\mathcal{A}$ of (19) satisfies $A_n \in \mathcal{I}_\mathcal{A}$, $n \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{I}_\mathcal{A}$, and $\mathcal{I}_\mathcal{A} \in \mathcal{X}$.

PROOF. Consider first the case $A_1 \subset A_2 \subset \cdots$. Since $\mathcal{I} \in \mathcal{X}$ implies that $\mathcal{I}$ is a chain containing $\emptyset$ and Ω, $\mathcal{I}_\mathcal{A}$ also shares these properties, because, for example, if $m < n$ then for $I, I' \in \mathcal{I}$,

$$(I \cup A_m) \cap A_{m+1} \subset A_{m+1} \subset A_n \subset (I' \cup A_n) \cap A_{n+1}.$$

To prove $\mathcal{I}_\mathcal{A} \in \mathcal{X}$, it only remains to prove $\sigma[\mathcal{I}_\mathcal{A}] = \mathcal{F}$.

Before proceeding with proving this, note that $\bigcup_{n\in\mathbb{N}} A_n = \emptyset \cup \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{I}_\mathcal{A}$, $A_1 = \Omega \cap A_1 \in \mathcal{I}_\mathcal{A}$, and for $n \in \mathbb{N}$, $A_{n+1} = (\Omega \cup A_n) \cap A_{n+1} \in \mathcal{I}_\mathcal{A}$,

Returning to the proof of $\sigma[\mathcal{I}_\mathcal{A}] = \mathcal{F}$, note that $\mathcal{I} \in \mathcal{X}$ implies $\sigma[\mathcal{I}] = \mathcal{F}$, hence it suffices to prove $\sigma[\mathcal{I}_\mathcal{A}] \supset \mathcal{I}$. To prove this, let $I \in \mathcal{I}$, and note that

$$I = (I \cap (\bigcup_{n\in\mathbb{N}} A_n)^c) \cup (I \cap A_1) \cup \bigcup_{n\in\mathbb{N}} (I \cap A_n^c \cap A_{n+1}).$$

Since $I \cup \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{I}_\mathcal{A}$ and $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{I}_\mathcal{A}$,

$$I \cap (\bigcup_{n\in\mathbb{N}} A_n)^c) = (I \cup \bigcup_{n\in\mathbb{N}} A_n) \cap (\bigcup_{n\in\mathbb{N}} A_n)^c \in \sigma[\mathcal{I}_\mathcal{A}],$$

and for $n \in \mathbb{N}$ since $(I \cup A_n) \cap A_{n+1} \in \mathcal{I}_\mathcal{A}$ and $A_n \in \mathcal{I}_\mathcal{A}$,

$$I \cap A_n^c \cap A_{n+1} = ((I \cup A_n) \cap A_{n+1}) \cap A_n^c \in \sigma[\mathcal{I}_\mathcal{A}].$$

Therefore $I \in \sigma[\mathcal{I}_\mathcal{A}]$, which proves $\mathcal{I} \subset \sigma[\mathcal{I}_\mathcal{A}]$.

Next let $A_1 \supset A_2 \supset \cdots$. All the claims except $\sigma[\mathcal{I}_\mathcal{A}] \supset \mathcal{I}$ are proved in a similar way as the previous case. Note that for $I \in \mathcal{I}$ we have

$$I = (I \cap \bigcap_{n\in\mathbb{N}} A_n) \cup (I \cap A_1^c) \cup \bigcup_{n\in\mathbb{N}} (I \cap A_{n+1}^c \cap A_n).$$

Since $I \cup A_1 \in \mathcal{I}_\mathcal{A}$ and $A_1 = \emptyset \cup A_1 \in \mathcal{I}_\mathcal{A}$,

$$I \cap A_1^c = (I \cup A_1) \cap A_1^c \in \sigma[\mathcal{I}_\mathcal{A}],$$

and for $n \in \mathbb{N}$ since $(I \cup A_{n+1}) \cap A_n \in \mathcal{I}_\mathcal{A}$ and $A_{n+1} \in \mathcal{I}_\mathcal{A}$,

$$I \cap A_{n+1}^c \cap A_n = ((I \cup A_{n+1}) \cap A_n) \cap A_{n+1}^c \in \sigma[\mathcal{I}_\mathcal{A}].$$

Therefore $I \in \sigma[\mathcal{I}_\mathcal{A}]$, which proves $\mathcal{I} \subset \sigma[\mathcal{I}_\mathcal{A}]$. □

Proof of Proposition 2.3. Let v be a non-decreasing continuous set function satisfying $v(\emptyset) = 0$. To prove (15), let $\mathcal{A} = \{A_n \mid n = 1, 2, 3, \ldots\} \subset \mathcal{F}$ be a sequence satisfying $A_1 \supset A_2 \supset \cdots$, and let $\mathcal{I}_\mathcal{A}$ be as in (19). Lemma 2.5 then implies $\mathcal{I}_\mathcal{A} \in \mathcal{X}$. Denote by $\mathcal{J}_\mathcal{A}$ the finitely additive class generated by $\mathcal{I}_\mathcal{A}$. Since v is continuous, Definition 2.1 implies that $\mu_{v,\mathcal{I}_\mathcal{A}} : \mathcal{J}_\mathcal{A} \to \mathbb{R}$ is continuous. Since Lemma 2.5 implies $A_n \in \mathcal{I}_\mathcal{A} \subset \mathcal{J}_\mathcal{A}$, $n \in \mathbb{N}$, and $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{I}_\mathcal{A} \subset \mathcal{J}_\mathcal{A}$, continuity of $\mu_{v,\mathcal{I}_\mathcal{A}}$ on $\mathcal{J}_\mathcal{A}$ implies $\lim_{n\to\infty} \mu_{v,\mathcal{I}_\mathcal{A}}(A_n) = \mu_{v,\mathcal{I}_\mathcal{A}}(\bigcap_{n\in\mathbb{N}} A_n)$. Also (8) implies $\mu_{v,\mathcal{I}_\mathcal{A}}(A_n) = v(A_n)$, $n \in \mathbb{N}$, and $\mu_{v,\mathcal{I}_\mathcal{A}}(\bigcap_{n\in\mathbb{N}} A_n) = v(\bigcap_{n\in\mathbb{N}} A_n)$. Therefore we have $\lim_{n\to\infty} v(A_n) = v(\bigcap_{n\in\mathbb{N}} A_n)$, which proves (15).

Proof of (14) is similar to that of (15), if we replace $A_1 \supset A_2 \supset \cdots$, $\bigcap_{n\in\mathbb{N}} A_n$, and (19) by $A_1 \subset A_2 \subset \cdots$, $\bigcup_{n\in\mathbb{N}} A_n$, and (18), respectively. □

Proof of Proposition 2.2. That the continuity of v implies (14) and (15) is a consequence of Proposition 2.3. To prove the converse, assume that v satisfies (3), (4), (14), and (15). Let $\mathcal{I} \in \mathcal{X}$ and $\mathcal{J}$ be the finitely additive class generated by $\mathcal{I}$. As in standard measure theory, to prove continuity of $\mu_{v,\mathcal{I}}$ on $\mathcal{J}$ it suffices to prove that for any sequence $A_n \in \mathcal{J}$, $n = 1, 2, 3, \ldots$, satisfying $A_1 \supset A_2 \supset \cdots$ and $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, $\lim_{n\to\infty} \mu_{v,\mathcal{I}}(A_n) = 0$. For such sequence $\{A_n\}$, (15) implies, with $v(\emptyset) = 0$, $\lim_{n\to\infty} v(A_n) = 0$. On the other hand Lemma 2.4 implies $\mu_{v,\mathcal{I}}(A_n) \leqq v(A_n)$, $n \in \mathbb{N}$. These with non-negativity of measure $\mu_{v,\mathcal{I}}$ imply $\lim_{n\to\infty} \mu_{v,\mathcal{I}}(A_n) = 0$, which completes the proof of continuity of v. □

Remark 2.6. In Proposition 2.2 both (14) and (15) are stated in order to make the statement also hold when 'submodular' is replaced by 'supermodular'. It is a known elementary fact that if v is non-decreasing and submodular then (15) implies (14).

Note that if v is non-decreasing submodular and (14) holds, (15) does not necessarily follow. In fact, if $\mu : \mathcal{F} \to \mathbb{R}$ is a finite measure such that there exists a sequence $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, such that $A_1 \supset A_2 \supset \cdots$, $\mu(A_n) > 0$, $n \in \mathbb{N}$, and $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, then $v : \mathcal{F} \to \mathbb{R}$

defined by $v(A) = \begin{cases} \mu(A)+1, & A \neq \emptyset, \\ 0 & A = \emptyset, \end{cases}$ is non-decreasing submodular and (14) holds, but (15) fails because

$$\lim_{n\to\infty} v(A_n) = \lim_{n\to\infty} \mu(A_n) + 1 = \mu(\bigcap_{n\in\mathbb{N}} A_n) + 1 = 1 \neq 0 = v(\bigcap_{n\in\mathbb{N}} A_n).$$

An explicit example is the one-dimensional Lebesgue measure $\mu = \mu_1$ on an interval $(\Omega, \mathcal{F}, \mu) = ((0,1], \mathcal{B}_1((0,1]), \mu_1)$ and $A_n = (0, \frac{1}{n}]$, $n \in \mathbb{N}$.

We move on to the main theorem of this section on the representation formula of submodular functions.

THEOREM 2.7. *Let $(\Omega, \mathcal{F})$ be a measurable space satisfying $\mathcal{X} \neq \emptyset$. Let $v : \mathcal{F} \to \mathbb{R}$ be a non-decreasing and continuous set function, satisfying $v(\emptyset) = 0$, where continuity of set function is as in Definition 2.1. Then the following (a), (b), (c), (d) are equivalent.*

(a) v is submodular, i.e., (3) holds.

(b) For all $\mathcal{I} \in \mathcal{X}$, the extremal measure $\mu_{v,\mathcal{I}}$ of v corresponding to $\mathcal{I}$ determined by (8) satisfies $\mu_{v,\mathcal{I}} \in \mathcal{C}_{-,v}(\Omega)$, where $\mathcal{C}_{-,v}(\Omega)$ is defined in (1) with $A = \Omega$.

(c) The representation (9) holds.

(d) For all $A, B \in \mathcal{F}$ satisfying $B \subset A$, (2) holds.

PROOF. We will prove (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (a) in this order.

(a) ⇒ (b). Let $\mathcal{I} \in \mathcal{X}$ and let $\mathcal{J}$ denote the finitely additive class generated by $\mathcal{I}$. Since $\Omega \in \mathcal{I}$, $\mu_{v,\mathcal{I}}(\Omega) = v(\Omega)$. Assumption (a) and Lemma 2.4 imply $\mu_{v,\mathcal{I}}(J) \leqq v(J)$, $J \in \mathcal{J}$. By the assumption that v is continuous, $\mu_{v,\mathcal{I}}$ is uniquely extended to a measure on $\mathcal{F}$, which we also write $\mu_{v,\mathcal{I}}$. According to standard measure theory, the measure $\mu_{v,\mathcal{I}}$ on $\mathcal{F} = \sigma[\mathcal{I}] = \sigma[\mathcal{J}]$ is approximated by the restriction of $\mu_{v,\mathcal{I}}$ to $\mathcal{J}$. Hence $\mu_{v,\mathcal{I}}(A) \leqq v(A)$ holds for all $A \in \mathcal{F}$, which further implies $\mu_{v,\mathcal{I}} \in \mathcal{C}_{-,v}(\Omega)$.

(b) ⇒ (c) . Fix $A \in \mathcal{F}$ arbitrarily. Since by assumption $\mathcal{X} \neq \emptyset$, there is $\mathcal{I} \in \mathcal{X}$. Then the insertion of A also satisfies $\mathcal{I}_A \in \mathcal{X}$. With $A \in \mathcal{I}_A$ we also have $\mu_{v,\mathcal{I}_A}(A) = v(A)$. Therefore we have $v(A) \leqq \sup_{\mathcal{I}\in\mathcal{X}} \mu_{v,\mathcal{I}}(A)$. On the other hand, by assumption (b) and the definition of $\mathcal{C}_{-,v}(\Omega)$, we have $\mu_{v,\mathcal{I}}(A) \leqq v(A)$ for all $\mathcal{I} \in \mathcal{X}$, which further implies $\sup_{\mathcal{I}\in\mathcal{X}} \mu_{v,\mathcal{I}}(A) \leqq v(A)$. Hence the equality holds. Since $A \in \mathcal{F}$ is arbitrary, (9) follows.

(c) ⇒ (d) . Let $A, B \in \mathcal{F}$ satisfy $B \subset A$. The definition of $\mathcal{C}_{-,v}(A)$ implies $\mu(B) \leqq v(B)$ for all $\mu \in \mathcal{C}_{-,v}(A)$. Hence to prove (2) it suffices to prove existence of $\mu \in \mathcal{C}_{-,v}(A)$ satisfying $v(B) = \mu(B)$.

Note that $B \subset A$ and let $\mathcal{I}_{A,B}$ the sequential insertion of A and B in (17). Assumption (c) then implies $\mu_{v,\mathcal{I}_{A,B}}(C) \leqq v(C)$ for all $C \in \mathcal{F}$ satisfying $C \subset A$. $A, B \in \mathcal{I}_{A,B}$ and (8) imply $\mu_{v,\mathcal{I}_{A,B}}(A) = v(A)$ and $\mu_{v,\mathcal{I}_{A,B}}(B) = v(B)$. Hence the restriction $\mu_{v,\mathcal{I}_{A,B}}|_A$ of $\mu_{v,\mathcal{I}_{A,B}}$ to A is in $\mathcal{C}_{-,v}(A)$, which is the measure we are looking for.

(d) ⇒ (a) This claim is essentially [5, Proposition 1], whose proof works here. Let $A, B \in \mathcal{F}$ and substitute A in (2) by $A \cup B$ to obtain $v(B) = \sup_{\mu \in \mathcal{C}_{-,v}(A \cup B)} \mu(B)$ and $v(A) = \sup_{\mu \in \mathcal{C}_{-,v}(A \cup B)} \mu(A)$, which further imply $v(B) \geqq \mu(B)$ and $v(A) \geqq \mu(A)$ for all $\mu \in \mathcal{C}_{-,v}(A \cup B)$. Also (1) implies $v(A \cup B) = \mu(A \cup B)$ for all $\mu \in \mathcal{C}_{-,v}(A \cup B)$. Finally, for $\epsilon > 0$, $v(A \cap B) = \sup_{\mu \in \mathcal{C}_{-,v}(A \cup B)} \mu(A \cap B)$ implies that there exists $\mu \in \mathcal{C}_{-,v}(A \cup B)$ such that $v(A \cap B) \leqq \mu(A \cap B) + \epsilon$. These equality and inequalities imply

$$\begin{aligned} &v(A \cup B) + v(A \cap B) - v(A) - v(B) \\ &\leqq \mu(A \cup B) + \mu(A \cap B) + \epsilon - \mu(A) - \mu(B) = \epsilon. \end{aligned}$$

ϵ can be any positive constant, hence (3) follows. □

In [5] we used the term continuous to mean (14) and (15) also for non-decreasing supermodular functions (convex games). As noted at the end of § 1, (11) implies that Proposition 2.2 holds also for supermodular functions. Furthermore, a convex game version of Theorem 2.7 also holds.

COROLLARY 2.8. *Let $(\Omega, \mathcal{F})$ be a measurable space satisfying $\mathcal{X} \neq \emptyset$. Let $v : \mathcal{F} \to \mathbb{R}$ be a non-decreasing and continuous set function, satisfying $v(\emptyset) = 0$. Then the following (a), (b), (c), (d) are equivalent.*

(a) v is supermodular (convex game), namely, $v(A) + v(B) \leqq v(A \cup B) + v(A \cap B)$, $A, B \in \mathcal{F}$, holds.

(b) For all $\mathcal{I} \in \mathcal{X}$, the extremal measure $\mu_{v,\mathcal{I}}$ of v corresponding to $\mathcal{I}$ determined by (8) satisfies $\mu_{v,\mathcal{I}} \in \mathcal{C}_{+,v}(\Omega)$, where $\mathcal{C}_{+,v}(\Omega) = \{\mu \in \mathcal{M}(\Omega) \mid \mu(\Omega) = v(\Omega),\ \mu(B) \geqq v(B),\ B \in \mathcal{F}\}$.

(c) It holds that $v(A) = \inf_{\mathcal{I} \in \mathcal{X}} \mu_{v,\mathcal{I}}(A),\ A \in \mathcal{F}$.

(d) For all $A, B \in \mathcal{F}$ satisfying $B \subset A$, $v(B) = \inf_{\mu \in \mathcal{C}_{+,v}(A)} \mu(B)$ holds.

3 Choquet integration and representation formula for submodular function

In this section, we fix a measurable space $(\Omega, \mathcal{F})$ which satisfies $\mathcal{X} \neq \emptyset$.

3.1 Choquet integration

Let $v : \mathcal{F} \to \mathbb{R}$ be a non-decreasing set function, satisfying $v(\emptyset) = 0$, and $f : \Omega \to \mathbb{R}$ a real valued measurable function.

Note that if we define $g : \mathbb{R} \to \mathbb{R}$ by $g(z) = v(\{\omega \in \Omega \mid f(\omega) > z\})$, and $g^{-1}(a) = \sup_{g(z)>a} z$ for $a \geqq 0$, non-decreasing property of v implies $\{z \in \mathbb{R} \mid g(z) > a\} = (-\infty, g^{-1}(a))$, hence g is a 1-dimensional Borel measurable function, and we can consider the standard 1-dimensional Lebesgue integration of g. We define

$$v(f) = \lim_{y \to -\infty} \left(y\, v(\Omega) + \int_y^\infty v(\{\omega \in \Omega \mid f(\omega) > z\})\, dz \right) \tag{20}$$

whenever the right-hand side is a real value. If either the Lebesgue integration or the limit diverges in the right-hand side of (20) we do not define $v(f)$. If $v(f)$ of (20) is defined it is equal to the asymmetric integral in terms of [4, Chap. 5]. We will refer to (20) as Choquet integration (of f with respect to v) in this paper.

If $f(\omega) \geqq x$, $\omega \in \Omega$, holds for an $x \in \mathbb{R}$, then

$$\int_y^x v(\{\omega \in \Omega \mid f(\omega) > z\}) dz = (x - y)\, v(\Omega), \quad y \leqq x,$$

hence in this case (20) has a simpler expression

$$v(f) = x\, v(\Omega) + \int_x^\infty v(\{\omega \in \Omega \mid f(\omega) > z\})\, dz. \tag{21}$$

Simple function, a measurable function whose image is a finite set, is an elementary example of Choquet integrable function. We find it convenient to adopt a representation

$$f = \sum_{i=1}^n a_i\, \mathbf{1}_{A_i}\,, \tag{22}$$

where

$$\begin{aligned} &a_i \geqq 0,\ i = 1, \ldots, n-1,\ a_n \in \mathbb{R}, \\ &\emptyset = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset A_n = \Omega. \end{aligned} \tag{23}$$

Here and in the following, we use a symbol $\mathbf{1}_A$ to denote a characteristic function of a set A defined by $\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$ (We use $\mathbf{1}$. on any space.)

We then have $A_i = \{\omega \in \Omega \mid f(\omega) \geqq \sum_{j=i}^{n} a_j\} \in \mathcal{F},\ i = 1, \ldots, n$, hence (21) with $x = a_n$ and (23) implies

$$v(f) = \sum_{i=1}^{n} a_i\, v(A_i). \tag{24}$$

Note that f of (22) takes values $b_i = \sum_{j=i}^{n} a_j\,,\ i = 1, \ldots, n$, with $b_1 \geqq b_2 \geqq \cdots \geqq b_n = a_n\,.$ With $B_i = A_i \cap A_{i-1}^c,\ i = 1, \ldots, n$, we have $B_i \cap B_j = \emptyset,\ i \neq j,\ \bigcup_{i=1}^{n} B_i = A_n = \Omega$, and $f = \sum_{i=1}^{n} b_i\, \mathbf{1}_{B_i}$, which perhaps is closer to a familiar expression of a simple function. With this expression, (24) implies $v(f) = \sum_{i=1}^{n-1} (b_i - b_{i+1})\, v(\bigcup_{j=1}^{i} B_j) + b_n v(\Omega)$.

3.2 Monotone convergene theorem

Hereafter we assume that $v :\ \mathcal{F} \to \mathbb{R}$ is a continuous and non-decreasing set function, We then have an analog of monotone convergence theorem for the Choquet integration with respect to v. To state the theorem, we extend some elementary terms in measure theory to continuous, non-decreasing set function v. We say that a measurable set $N \in \mathcal{F}$ is a v-null set, if

$$v(N \cup A) = v(N^c \cap A) = v(A),\ A \in \mathcal{F}. \tag{25}$$

For a sequence of measurable functions $f_n :\ \Omega \to \mathbb{R},\ n \in \mathbb{N}$, and a measurable function $f :\ \Omega \to \mathbb{R}$, we say that the sequence converges v-almost surely to f, if there exists a v-null set $N \in \mathcal{F}$ such that

$$\lim_{n\to\infty} f_n(\omega) = f(\omega),\ \omega \in N^c. \tag{26}$$

THEOREM 3.1. *Let $(\Omega, \mathcal{F})$ be a measurable space satisfying $\mathcal{X} \neq \emptyset$, and $v :\ \mathcal{F} \to \mathbb{R}$ be a non-decreasing, continuous set function, satisfying $v(\emptyset) = 0$.*

If $f_n :\ \Omega \to \mathbb{R},\ n \in \mathbb{N}$, and $f :\ \Omega \to \mathbb{R}$ are measurable and Choquet integrable functions such that f_n are pointwise non-decreasing in n and $f = \lim_{n\to\infty} f_n$, v-almost surely, then $\lim_{n\to\infty} v(f_n) = v(f)$ holds.

PROOF. First we assume that f_1 is non-negative valued. Then $f_n\,,\ n \in \mathbb{N}$, and f are also non-negative. For $z \in \mathbb{R}$, put $C_n(z) = \{\omega \in \Omega \mid f_n(\omega) > z\},\ n \in \mathbb{N}$, and $\mathcal{C}(z) = \{C_n(z) \mid n \in \mathbb{N}\}$. By assumption that f_n is pointwise non-dereasing in n, the

limit $\lim_{n\to\infty} f_n : \Omega \to \mathbb{R} \cup \{+\infty\}$ exists if we conventionally allow $+\infty$ for a limit value. Furthermore it holds that $C_1(z) \subset C_2(z) \subset \cdots$ and $\bigcup_{n\in\mathbb{N}} C_n(z) = \{\omega \in \Omega \mid \lim_{n\to\infty} f_n(\omega) > z\}$. Define the non-negative functions $g_n : \mathbb{R} \to \mathbb{R}_+$, $n \in \mathbb{N}$, and $g : \mathbb{R} \to \mathbb{R}_+$ by $g_n(z) = v(C_n(z))$, $n \in \mathbb{N}$, and $g(z) = v(\bigcup_{n\in\mathbb{N}} C_n(z))$. Since by assumption v is non-decreasing g_n is pointwise non-decreasing in n and $g_n(z) \leqq g(z)$, $z \in \mathbb{R}$.

By assumption $\mathcal{X} \neq \emptyset$ there exists $\mathcal{I} \in \mathcal{X}$, hence Lemma 2.5 implies $\mathcal{I}_{\mathcal{C}(z)} \in \mathcal{X}$. Therefore, assumption of continuity of v implies that there exists a measure $\mu_{v,\mathcal{I}_{\mathcal{C}(z)}} : \mathcal{F} \to \mathbb{R}_+$ defined by (8) with $\mathcal{I} = \mathcal{I}_{\mathcal{C}(z)}$. It therefore follows from continuity of measure that

$$\begin{aligned}\lim_{n\to\infty} g_n(z) &= \lim_{n\to\infty} v(C_n(z)) = \lim_{n\to\infty} \mu_{v,\mathcal{I}_{\mathcal{C}(z)}}(C_n(z)) = \mu_{v,\mathcal{I}_{\mathcal{C}(z)}}(\bigcup_{n\in\mathbb{N}} C_n(z)) = v(\bigcup_{n\in\mathbb{N}} C_n(z)) \\ &= g(z),\ z \in \mathbb{R}.\end{aligned}$$

For $z \in \mathbb{R}$, put $C(z) = \{\omega \in \Omega \mid f(\omega) > z\}$ and $N = \{\omega \in \Omega \mid f(\omega) \neq \lim_{n\to\infty} f_n(\omega)\}$, Then $\bigcup_{n\in\mathbb{N}} C_n(z) \cap N^c \subset C(z) \subset \bigcup_{n\in\mathbb{N}} C_n(z) \cup N$ hols. Assumption of v-almost sure onvergence implies that N is a v-null set, hence we have $v(C(z)) = v(\bigcup_{n\in\mathbb{N}} C_n(z)) = g(z)$.

Choquet integrability assumptions of f_n and f, with (21) with $x = 0$ then imply

$$v(f) = \int_0^\infty g(z)\,dz, \quad v(f_n) = \int_0^\infty g_n(z)\,dz,\ n \in \mathbb{N}.$$

Hence the standard measure theoretic monotone convergence theorem for 1-dimensional Lebesgue measure implies $\lim_{n\to\infty} v(f_n) = \lim_{n\to\infty} \int_0^\infty g_n(z)\,dz = \int_0^\infty g(z)\,dz = v(f)$, which proves the claim when the functions are non-negative.

Next, we consider the case that the functions are bounded from below, and assume that there exists $y_0 \in \mathbb{R}$ such that $f_1(\omega) \geqq y_0$ $\omega \in \Omega$. By pointwise non-decreasing properties of the functions, it follows that $f_n(\omega) \geqq y_0$ $n \in \mathbb{N}$, and $f(\omega) \geqq y_0$, for all $\omega \in \Omega$. Subtracting the constant y_0 from the functions, put $\tilde{f}_n = f_n - y_0$, $n \in \mathbb{N}$, and $\tilde{f} = f - y_0$. Then $\tilde{f}_n$, $n \in \mathbb{N}$, and $\tilde{f}$ are non-negative vauled functions satisfying the assumptions of the Theorem, hence the proof in the previous paragraphs implies $\lim_{n\to\infty} v(\tilde{f}_n) = v(\tilde{f})$, v-almost surely. Also (21) implies, with change of integration variable $z' = z - y_0$,

$$\begin{aligned}v(f) &= y_0\, v(\Omega) + \int_{y_0}^\infty v(\{\omega \in \Omega \mid f(\omega) > z\})\,dz \\ &= y_0\, v(\Omega) + \int_0^\infty v(\{\omega \in \Omega \mid \tilde{f}(\omega) > z'\})\,dz' = y_0\, v(\Omega) + v(\tilde{f}),\end{aligned}$$

and similar formula for f_n, $n \in \mathbb{N}$. Therefore we have $\lim_{n\to\infty} v(f_n) = v(f)$, v-almost surely.

Finally we consider the general case. Since v is non-decreasing, (20) implies that for any $\epsilon > 0$ there exists $y_0 \in \mathbb{R}$ such that

$$v(f_1) \leqq y_0\, v(\Omega) + \int_{y_0}^{\infty} v(\{\omega \in \Omega \mid f_1(\omega) > z\})\, dz \leqq v(f_1) + \epsilon. \tag{27}$$

Round up the values of the functions smaller than y_0 and put $\tilde{f}_n = f_n \vee y_0$, $n \in \mathbb{N}$, and $\tilde{f} = f \vee y_0$, where $a \vee b = a$ if $a \geqq b$ and otherwise $a \vee b = b$. Then $\tilde{f}_n$, $n \in \mathbb{N}$, and $\tilde{f}$ are bounded from below and satisfy the assumptions of the Theorem, hence the proof in the previous paragraph implies

$$\lim_{n\to\infty} v(\tilde{f}_n) = v(\tilde{f}), \tag{28}$$

v-almost surely. Also $\tilde{f}_n = f_n \vee y_0$ and (21) imply

$$\begin{aligned} v(\tilde{f}_n) &= y_0\, v(\Omega) + \int_{y_0}^{\infty} v(\{\omega \in \Omega \mid \tilde{f}_n(\omega) > z\})\, dz \\ &= y_0\, v(\Omega) + \int_{y_0}^{\infty} v(\{\omega \in \Omega \mid f_n(\omega) > z\})\, dz,\ n \in \mathbb{N}, \end{aligned} \tag{29}$$

hence (27) implies

$$0 \leqq v(\tilde{f}_1) - v(f_1) \leqq \epsilon. \tag{30}$$

Also, (29) and (20) imply

$$v(\tilde{f}_n) - v(f_n) = \lim_{y\to-\infty} \left((y_0 - y)\, v(\Omega) - \int_{y}^{y_0} v(\{\omega \in \Omega \mid f_n(\omega) > z\})\, dz \right),\ n \in \mathbb{N}, \tag{31}$$

Since f_n is pointwise non-decreasing in n and since v is non-decreasing $v(\{\omega \in \Omega \mid f_n(\omega) > z\})$ is also non-decreasing in n and is no larger than $v(\Omega)$. Therefore (31) further implies

$$v(\tilde{f}_1) - v(f_1) \geqq v(\tilde{f}_2) - v(f_2) \geqq \cdots \geqq \lim_{n\to\infty} (v(\tilde{f}_n) - v(f_n)) \geqq 0 \tag{32}$$

The argument between (29) and (32), applied to the function f and $\tilde{f}$ implies

$$v(\tilde{f}) - v(f) = \lim_{y\to-\infty} \left((y_0 - y)\, v(\Omega) - \int_{y}^{y_0} v(\{\omega \in \Omega \mid f(\omega) > z\})\, dz \right) \geqq 0. \tag{33}$$

Combining (30), (32), (28), and (33), we therefore have

$$\epsilon \geqq \lim_{n\to\infty} v(\tilde{f}_n) - \lim_{n\to\infty} v(f_n) = v(\tilde{f}) - \lim_{n\to\infty} v(f_n) \geqq v(f) - \lim_{n\to\infty} v(f_n). \tag{34}$$

Since by assumption f_n is pointwise non-decreasing in n and converges v-almost surely to f, $v(f_n) \leqq v(f)$, hence $\lim_{n\to\infty} v(f_n) \leqq v(f)$. Since $\epsilon > 0$ is arbitrary $\lim_{n\to\infty} v(f_n) = v(f)$ follows. □

As in standard measure theory, a simple function approximation of a non-negative Choquet integrable function $f:\ \Omega \to \mathbb{R}_+$ defined by

$$f_k := \sum_{i=1}^{k\,2^k} 2^{-k}(i-1)\,\mathbf{1}_{2^{-k}(i-1)<f\leqq 2^{-k}i} + k\,\mathbf{1}_{\{\omega\in\Omega | f(\omega)>k\}},\quad k\in\mathbb{N}, \tag{35}$$

is an example of non-decreasing sequence of functions which converges pointwise to f as $k\to\infty$. Theorem 3.1 then implies $v(f)=\lim_{k\to\infty} v(f_k)$.

3.3 Representation formula for Choquet integration and coherent risk measure

The representation (9) for submodular functions in the framework of Theorem 2.7 implies corresponding formula for Choquet integrations.

THEOREM 3.2. *Let $(\Omega,\mathcal{F})$ be a measurable space satisfying $\mathcal{X}\neq\emptyset$. Let $v:\ \mathcal{F}\to\mathbb{R}$ be a non-decreasing and continuous submodular function satisfying $v(\emptyset)=0$. Then for a bounded measurable function $f:\ \Omega\to\mathbb{R}$*

$$v(f)=\sup_{\mathcal{I}\in\mathcal{X}}\int_\Omega f(\omega)\,\mu_{v,\mathcal{I}}(d\omega) \tag{36}$$

holds.

PROOF. Since f is bounded and measurable, f is both v-integrable and integrable with respect to (finite) measures. Also since (21) implies $v(f+a)=av(\Omega)+v(f)$, we may assume from beginning that f is non-negative in the proof of (36) by adding a positive constant to f. Let f_k, $k\in\mathbb{N}$, be the series of simple function approximations (35) of f. For simplicity of notation we follow that of (23) and write (35) as

$$\begin{aligned} &f_k=\sum_{i=1}^{n_k} a_i^{(k)}\,\mathbf{1}_{A_i^{(k)}},\quad a_i^{(k)}>0,\ i=1,\dots,n_k-1,\ a_{n_k}^{(k)}=0,\\ &A_1^{(k)}\subset A_2^{(k)}\subset\cdots\subset A_{n_k}^{(k)}=\Omega,\quad \mathcal{A}_k=\{A_i^{(k)}\mid i=1,\dots,n_k\}. \end{aligned}$$

Note that we can put $a_{n_k}^{(k)}=0$ because we assume that f is non-negative. Then (24) implies $v(f_k)=\sum_{i=1}^{n_k} a_i^{(k)}\,v(A_i^{(k)})$.

As noted below (13), the definition of the continuity of v implies that (8) uniquely defines a measure $\mu_{v,\mathcal{I}}:\ \mathcal{F}\to\mathbb{R}$, for each $\mathcal{I}\in\mathcal{X}$. In particular, for $\mathcal{I}_{\mathcal{A}_k}\in\mathcal{X}$, the insertion of a chain $\mathcal{A}_k$ to $\mathcal{I}\in\mathcal{X}$ we have, from (8),

$$v(f_k)=\sum_{i=1}^{n_k} a_i^{(k)}v(A_i^{(k)})=\sum_{i=1}^{n_k} a_i^{(k)}\mu_{v,\mathcal{I}_{\mathcal{A}_k}}(A_i^{(k)})=\int_\Omega f_k(\omega)\,\mu_{v,\mathcal{I}_{\mathcal{A}_k}}(d\omega). \tag{37}$$

Also, Lemma 2.4 implies, as in the proof of (a) to (b) in Theorem 2.7, $\mu_{v,\mathcal{I}}(A) \leqq v(A)$, $A \in \mathcal{F}$, $\mathcal{I} \in \mathcal{X}$. Therefore,

$$v(f_k) = \sum_{i=1}^{n_k} a_i^{(k)} v(A_i^{(k)}) \geqq \sum_{i=1}^{n_k} a_i^{(k)} \mu_{v,\mathcal{I}}(A_i^{(k)}) = \int_\Omega f_k(\omega)\, \mu_{v,\mathcal{I}}(d\omega). \ \mathcal{I} \in \mathcal{X}. \tag{38}$$

Since f_k is pointwise non-decreasing in k,

$$\int_\Omega f_k(\omega)\, \mu_{v,\mathcal{I}}(d\omega) \leqq \int_\Omega f(\omega)\, \mu_{v,\mathcal{I}}(d\omega),\ \mathcal{I} \in \mathcal{X},\ \text{and } v(f_k) \leqq v(f), \quad k \in \mathbb{N}. \tag{39}$$

The standard monotone convergence theorem for measures implies

$$(\forall \epsilon > 0)(\forall \mathcal{I} \in \mathcal{X})\exists k_0 \in \mathbb{N};\ (\forall k \geqq k_0)\ \int_\Omega f(\omega)\, \mu_{v,\mathcal{I}}(d\omega) \leqq \int_\Omega f_k(\omega)\, \mu_{v,\mathcal{I}}(d\omega) + \epsilon \tag{40}$$

and Theorem 3.1 implies

$$(\forall \epsilon > 0)\exists k_0 \in \mathbb{N};\ (\forall k \geqq k_0)\ v(f) \leqq v(f_k) + \epsilon. \tag{41}$$

Combining (38), (39), and (40), we see that $v(f) \geqq \int_\Omega f(\omega)\, \mu_{v,\mathcal{I}}(d\omega) - \epsilon$ for any $\mathcal{I} \in \mathcal{X}$ and $\epsilon > 0$. Hence

$$v(f) \geqq \sup_{\mathcal{I} \in \mathcal{X}} \int_\Omega f(\omega)\, \mu_{v,\mathcal{I}}(d\omega). \tag{42}$$

Similarly, combining (37), (39), and (41), we see that for any $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that if $k \geqq k_0$ then $v(f) \leqq \int_\Omega f(\omega)\, \mu_{v,\mathcal{I}_{\mathcal{A}_k}}(d\omega) + \epsilon \leqq \sup_{\mathcal{I} \in \mathcal{X}} \int_\Omega f(\omega)\, \mu_{v,\mathcal{I}}(d\omega) + \epsilon$. Hence $v(f) \leqq \sup_{\mathcal{I} \in \mathcal{X}} \int_\Omega f(\omega)\, \mu_{v,\mathcal{I}}(d\omega)$, which, with (42), implies (36). □

Corollary 3.3. *Let $(\Omega, \mathcal{F})$ be a measurable space satisfying $\mathcal{X} \neq \emptyset$. Let $v : \mathcal{F} \to \mathbb{R}$ be a non-decreasing and continuous submodular function satisfying $v(\emptyset) = 0$, and define a function ρ on a set of bounded measurable functions by $\rho(f) = \dfrac{v(-f)}{v(\Omega)}$. Then the following hold.*

non-negativity: *If $f \geqq 0$ then $\rho(f) \leqq 0$,*

subadditivity: $\rho(f + g) \leqq \rho(f) + \rho(g)$,

positive homogeneity: *If $\lambda \geqq 0$ then $\rho(\lambda f) = \lambda \rho(f)$,*

translational invariance: *If $a \in \mathbb{R}$ then $\rho(f + a) = \rho(f) - a$.*

PROOF. The claims are straightforward consequences of (36) in Theorem 3.2. For example,

$$\begin{aligned}\rho(f+g) &= -\frac{1}{v(\Omega)} \inf_{\mathcal{I}\in\mathcal{X}} \int_\Omega (f(\omega)+g(\omega))\,\mu_{v,\mathcal{I}}(d\omega) \\ &\leqq -\frac{1}{v(\Omega)} \inf_{\mathcal{I}\in\mathcal{X}} \int_\Omega f(\omega)\,\mu_{v,\mathcal{I}}(d\omega) - \frac{1}{v(\Omega)} \inf_{\mathcal{I}\in\mathcal{X}} \int_\Omega g(\omega))\,\mu_{v,\mathcal{I}}(d\omega) = \rho(f)+\rho(g),\end{aligned}$$

which proves subadditivity. Proofs of non-negativity, positive homogeneity, and translational invariance are easier. □

In the field of mathematical finance, the set of properties in Corollary 3.3 is known to be the definition that ρ is a coherent risk measure [1, 2, 6, 7]. Thus Corollary 3.3 relates the Choquet integration with respect to a sub-modular set function to a coherent risk measure. This observation motivates reformulating in our framework the results found in mathematical finance, which we consider in the next section § 4.

4 Choquet integration for uniform case and law invariant coherent risk measure

In [6], coherent risk measure is studied on probability spaces. Given a probability measure P, the law invariance of a set function v means that $\mathrm{P}[\,A\,] = \mathrm{P}[\,B\,]$ implies $v(A) = v(B)$. In our approach, we started with a measurable space $(\Omega, \mathcal{F})$ without probability measure, hence we are free to intruduce a (finite) measure $\nu : \mathcal{F} \to \mathbb{R}$ and define law invariance as $\nu(A) = \nu(B)$ implies $v(A) = v(B)$.

In addition, it is assumed in [6] that $(\Omega, \mathcal{F}, \mathrm{P})$ is a standard probability space and that P is non-atomic, and with these assumptions, the proof is reduced to the 1-dimensional Borel measurable space on an interval $(\Omega, \mathcal{F}) = ([0,1), \mathcal{B}_1([0,1)))$, and P specified as the 1-dimensional Lebesgue measure. Here, we keep the only assumption $\mathcal{X} \neq \emptyset$, existence of a chain generating $\mathcal{F}$, for $(\Omega, \mathcal{F})$, and see how a formula corresponding to that studied in [6] is deduced in our framework.

As a simple example of submodular function whose dependence on the variable $A \in \mathcal{F}$ is given through $\nu(A)$, we note the follwing. In the following, for $a, b \in \mathbb{R}$ such that $a \leqq b$ we write $a \wedge b = a$ and $a \vee b = b$. For example, $a + b = (a \vee b) + (a \wedge b)$ and $a \vee b \geqq a \wedge b$ hold.

PROPOSITION 4.1. *Let $(\Omega, \mathcal{F})$ be a measurable space, $\nu : \mathcal{F} \to \mathbb{R}$ a measure on the space, and $c \in \mathbb{R}$. Then the set function $v : \mathcal{F} \to \mathbb{R}$ defined by $v(A) = c \wedge \nu(A)$, $A \in \mathcal{F}$, is submodular, i.e., satisfies (3).*

PROOF. Let $A, B \in \mathcal{F}$. Then additivity, non-negativity, and monotonicity of the measure ν imply

$$
\begin{aligned}
&v(A)+v(B)-v(A\cup B)-v(A\cap B)\\
&=\begin{cases}
c+c-c-c=0, & c\leqq \nu(A\cap B),\\
c+c-\nu(A\cap B)-c\geqq 0, & \nu(A\cap B)<c\leqq \nu(A)\wedge\nu(B),\\
(\nu(A)\wedge\nu(B))+c-\nu(A\cap B)-c\geqq 0, & \nu(A)\wedge\nu(B)<c\leqq \nu(A)\vee\nu(B),\\
\nu(A)+\nu(B)-\nu(A\cap B)-c & \\
=\nu(A\cup B)-c\geqq 0, & \nu(A)\vee\nu(B)<c\leqq \nu(A\cup B),\\
\nu(A)+\nu(B)-\nu(A\cup B)-\nu(A\cap B)=0, & \nu(A\cup B)\leqq c.
\end{cases}
\end{aligned}
$$

which proves (3). □

Submodularlity (3) is preserved by summation and multiplication of positive reals. This leads to considering v of a form $v(A)=\int_{\mathbb{R}_+} g(z)\wedge\nu(A)\,dz$ for some non-negative function g. If we choose the parameter z so that g is monotone, we could consider g as a distribution function of a measure. These considerations suggest considering a following form for a submodular set function v whose depedence on variable $A\in\mathcal{F}$ enters through $\nu(A)$.

THEOREM 4.2. *Let $(\Omega,\mathcal{F})$ be a measurable space satisfying $\mathcal{X}\neq\emptyset$, and $\nu\in\mathcal{M}(\Omega)$. For a finite measure $\mu\in\mathcal{M}(\Omega)$ satisfying $\mu\ll\nu$, denote the Radon–Nykodim derivative by $\frac{d\,\mu}{d\nu}:\ \Omega\to\mathbb{R}_+$, and define the distribution function $F_\mu:\ \mathbb{R}_+\to\mathbb{R}_+$ by*

$$
F_\mu(y)=\nu(\{\omega\in\Omega\mid \frac{d\,\mu}{d\nu}(\omega)\leqq y\}),\ y\in\mathbb{R}_+, \tag{43}
$$

and define a set function $v_\mu:\ \mathcal{F}\to\mathbb{R}$ by

$$
v_\mu(A)=\int_0^\infty (\nu(\Omega)-F_\mu(z))\wedge\nu(A)\,dz,\ A\in\mathcal{F}. \tag{44}
$$

Then the following hold.

(i) v_μ is non-decreasing, continuous, submodular, and satisfies $v_\mu(\emptyset)=0$.

(ii) It holds that

$$
v_\mu(\{\omega\in\Omega\mid \frac{d\,\mu}{d\nu}(\omega)>y\})=\mu(\{\omega\in\Omega\mid \frac{d\,\mu}{d\nu}(\omega)>y\}),\ y\in\mathbb{R}_+. \tag{45}
$$

(iii) For a non-negative valued measurable function $f:\ \Omega\to\mathbb{R}_+$, let $v_\mu(f)$ denote the Choquet integration of f with respect to v_μ in (21). Then

$$
v_\mu(f)\geqq\sup\{\int_\Omega f(\omega)\mu'(d\omega)\mid \mu'\in\mathcal{M}(\Omega),\ \mu'\ll\nu,\ F_{\mu'}=F_\mu\}, \tag{46}
$$

holds. In particular, it holds that

$$v_\mu(A) \geqq \sup\{\mu'(A) \mid \mu' \in \mathcal{M}(\Omega),\ \mu' \ll \nu,\ F_{\mu'} = F_\mu\},\ A \in \mathcal{F}. \tag{47}$$

PROOF. (i) Since ν is a measure, $\nu(\emptyset) = 0$, hence $v_\mu(\emptyset) = 0$. Non-decreasing property is obvious from the same property for measures and that integration preserves inequality. To prove that v_μ is submodular, substitute c in Proposition 4.1 with $\nu(\Omega) - F_\mu(z)$ and integrate over z, we see from Proposition 4.1 that v_μ satisfies (3).

To prove continuity, since we have shown that v_μ is non-decreasing and submodular, Proposition 2.2 now implies that it suffices to prove (14) and (15) for $v = v_\mu$. Since ν is a measure, (14) and (15) hold for $v = \nu$. This and monotone convergence theorem applied to (44) imply (14) and (15) for $v = v_\mu$.

(ii) Note a basic formula

$$\int_0^\infty \nu(f > z)\, dz := \int_0^\infty \nu(\{\omega \in \Omega \mid f(\omega) > z\})\, dz = \int_\Omega f\, d\nu, \tag{48}$$

valid for any non-negative measurable function $f : \Omega \to \mathbb{R}_+$. In particular, for v_μ as in (44), we have $v_\mu(\Omega) = \displaystyle\int_\Omega \frac{d\,\mu}{d\nu} d\nu = \mu(\Omega)$. By replacement $f \mapsto f \vee a$, (48) also implies that for any non-negative measurable function $f : \Omega \to \mathbb{R}_+$ and a non-negative constant $a \geqq 0$,

$$\begin{aligned} \int_a^\infty \nu(f > z)\, dz &= \int_0^\infty \nu(f \vee a > z)\, dz - a\nu(\Omega) = \int_\Omega f \vee a\, d\nu - a\nu(\Omega) \\ &= \int_{f \geqq a} f\, d\nu - a\nu(f \geqq a) = \int_{f > a} f\, d\nu - a\nu(f > a). \end{aligned} \tag{49}$$

Using (49) with $f = \dfrac{d\,\mu}{d\nu}$ and $a = y$ in (44) we have

$$\begin{aligned} &v_\mu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y\}) \\ &= \int_y^\infty \nu((\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > z\})\, dz + y\,\nu((\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y\}) \\ &= \int_{\frac{d\,\mu}{d\nu}(\omega) > y} \frac{d\,\mu}{d\nu}(\omega)\, \nu(d\omega) = \mu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y\} \end{aligned}$$

which proves (45).

(iii) Let $\mu' : \mathcal{F} \to \mathbb{R}_+$ be a measure satisfying $\mu' \ll \nu$ and $F_{\mu'} = F_\mu$. The definition (21) of Choquet integration, an elementary inequality $\nu(A) \wedge \nu(B) \geqq \nu(A \cap B)$, valid for

any $A, B \in \mathcal{F}$, and (48) then imply

$$\begin{aligned}
v_\mu(f) &= \int_0^\infty v_\mu(\{\omega \in \Omega \mid f(\omega) > z\})\, dz \\
&= \int_0^\infty \int_0^\infty (\nu(\Omega) - F_\mu(y)) \wedge \nu(\{\omega \in \Omega \mid f(\omega) > z\})\, dy\, dz \\
&= \int_0^\infty \int_0^\infty (\nu(\Omega) - F_{\mu'}(y)) \wedge \nu(\{\omega \in \Omega \mid f(\omega) > z\})\, dy\, dz \\
&= \int_0^\infty \int_0^\infty (\nu(\{\omega \in \Omega \mid \frac{d\,\mu'}{d\nu}(\omega) > y\})) \wedge \nu(\{\omega \in \Omega \mid f(\omega) > z\})\, dy\, dz \\
&\geqq \int_0^\infty \int_0^\infty (\nu(\{\omega \in \Omega \mid \frac{d\,\mu'}{d\nu}(\omega) > y\}) \cap \{\omega \in \Omega \mid f(\omega) > z\})\, dy\, dz \\
&= \int_0^\infty \int_\Omega \int_0^\infty \mathbf{1}_{\frac{d\,\mu'}{d\nu}(\omega) > y}\, \mathbf{1}_{f(\omega)>z}\, dy\, \nu(d\omega)\, dz \\
&= \int_0^\infty \int_\Omega \mathbf{1}_{f(\omega)>z}\, \frac{d\,\mu'}{d\nu}(\omega)\, \nu(d\omega)\, dz \\
&= \int_0^\infty \int_\Omega \mathbf{1}_{f(\omega)>z}\, \mu'(d\omega)\, dz = \int_0^\infty \mu'(\{\omega \in \Omega \mid f(\omega) > z\})\, dz \\
&= \int_\Omega f(\omega)\, \mu'(d\omega),
\end{aligned}$$

which proves (46).

By choosing $f = \mathbf{1}_A$ in (46), (47) follows.

□

As in [6, Theorem 7], we assume comonotonicity for further results. We say that the functions $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ are comonotone if $\{\{\omega \in \Omega \mid f(\omega) \leqq z\} \mid z \in \mathbb{R}\} \cup \{\{\omega \in \Omega \mid g(\omega) \leqq z\} \mid z \in \mathbb{R}\}$ is a chain, i.e., for each $(y, z) \in \mathbb{R}^2$, either $\{\omega \in \Omega \mid g(\omega) \leqq y\} \supset \{\omega \in \Omega \mid f(\omega) \leqq z\}$ or $\{\omega \in \Omega \mid g(\omega) \leqq y\} \subset \{\omega \in \Omega \mid f(\omega) \leqq z\}$ holds.

THEOREM 4.3. *Let $(\Omega, \mathcal{F})$ be a measurable space satisfying $\mathcal{X} \neq \emptyset$, and $\nu \in \mathcal{M}(\Omega)$. For a finite measure $\mu \in \mathcal{M}(\Omega)$ satisfying $\mu \ll \nu$, define a set function $v_\mu : \mathcal{F} \to \mathbb{R}$ by (44). For $y \geqq 0$ put $I_{\mu,y} = \{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y\}$ and put $\mathcal{I}_\mu = \{\emptyset, \Omega\} \cup \{I_{\mu,y} \mid y \in \mathbb{R}_+\}$ and assume that*

$$\mathcal{I}_\mu \in \mathcal{X} \text{ and } \operatorname{Im} F_\mu \supset \operatorname{Im} \nu, \tag{50}$$

where Im *denotes the image of the map.*

Then if a non-negative valued measurable function $f : \Omega \to \mathbb{R}_+$ is comonotone with $\frac{d\,\mu}{d\nu}$, the equality in (46) is attained by μ. Namely,

$$v_\mu(f) = \int_\Omega f(\omega)\, \mu(d\omega) = \sup\{\int_\Omega f(\omega)\mu'(d\omega) \mid \mu' \in \mathcal{M}(\Omega),\ \mu' \ll \nu,\ F_{\mu'} = F_\mu\} \tag{51}$$

holds.

PROOF. Note first that by assumptions there exists non-decreasing function $y_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned} &\nu(\{\omega \in \Omega \mid f(\omega) > y\}) = \nu(\Omega) - F_\mu(y_0(y)) \\ &= \nu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y_0(y)\}), \quad y \in \mathbb{R}_+. \end{aligned} \tag{52}$$

The definiton (44) of v_μ, (52), and (45) in Theorem 4.2 imply

$$\begin{aligned} &v_\mu(\{\omega \in \Omega \mid f(\omega) > y\}) = \int_0^\infty (\nu(\Omega) - F_\mu(z)) \wedge \nu(\{\omega \in \Omega \mid f(\omega) > y\})\, dz \\ &= \int_0^\infty (\nu(\Omega) - F_\mu(z)) \wedge \nu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y_0(y)\})\, dz \\ &= v_\mu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y_0(y)\}) \\ &= \mu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y_0(y)\}),\ y \in \mathbb{R}_+. \end{aligned} \tag{53}$$

Denote the set difference by $\triangle$, so that $A \triangle B = (A \cap B^c) \cup (A^c \cap B)$. By the comonotonicity assumption, either $\{\omega \in \Omega \mid f(\omega) > y\} \subset \{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y_0(y)\}$ or $\{\omega \in \Omega \mid f(\omega) > y\} \supset \{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y_0(y)\}$, and in either case (52) implies that the difference is measure 0. Hence

$$\nu(\{\omega \in \Omega \mid f(\omega) > y\} \triangle \{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y_0(y)\}) = 0, \quad y \in \mathbb{R}_+.$$

By assumption $\mu \ll \nu$ we then have $\mu(\{\omega \in \Omega \mid f(\omega) > y\} \triangle \{\omega \in \Omega \mid \frac{d\,\mu}{d\mu}(\omega) > y_0(y)\}) = 0$, which further implies $\mu(\{\omega \in \Omega \mid f(\omega) > y\}) = \mu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\mu}(\omega) > y_0(y)\})$, $y \in \mathbb{R}_+$. Substituting this in (53) we have

$$v_\mu(\{\omega \in \Omega \mid f(\omega) > y\}) = \mu(\{\omega \in \Omega \mid f(\omega) > y\}),\ y \in \mathbb{R}_+.$$

Using this in the definition (21) of Choquet integration and using (48) we arrive at

$$\begin{aligned} v_\mu(f) &= \int_0^\infty v_\mu(\{\omega \in \Omega \mid f(\omega) > y\})\, dy \\ &= \int_0^\infty \mu(\{\omega \in \Omega \mid f(\omega) > y\})\, dy = \int_\Omega f(\omega)\, \mu(d\omega). \end{aligned}$$

This with (46) in Theorem 4.2 implies (51). □

Theorem 4.3 corresponds to [6, Theorem 7] but the notation apparently is quite different. Before closing this section, we briefly look into the correspondence of notation with the reference, for convenience. As noted in Corollary 3.3 in the previous section, $v(f)$ corresponds to $\rho(-X)$ in [6]. The right continuous inverse of a distribution function F (in the sense of (54) below) is denoted by $Z(x, F)$ in the same reference. v_μ has an expression using the inverse of the distribution function (43).

PROPOSITION 4.4. *Let $(\Omega, \mathcal{F})$ be a measurable space satisfying $\mathcal{X} \neq \emptyset$, and $\nu \in \mathcal{M}(\Omega)$. As in Theorem 4.2, let $\mu \in \mathcal{M}(\Omega)$ satisfying $\mu \ll \nu$ be a finite measure whose Radon–Nykodim derivative is $\frac{d\,\mu}{d\nu} : \Omega \to \mathbb{R}_+$, and $F_\mu : \mathbb{R}_+ \to \mathbb{R}_+$ be its distribution function (43), and $v_\mu : \mathcal{F} \to \mathbb{R}$ defined by (44). Define F_μ^{-1}, the right continuous inverse function of F_μ, by*

$$F_\mu^{-1}(\beta) = \inf\{z \in \mathbb{R}_+ \mid F_\mu(z) > \beta\},\ \beta \in \mathbb{R}_+. \tag{54}$$

Then v_μ has an expression

$$v_\mu(A) = \int_{\nu(A^c)}^{\nu(\Omega)} F_\mu^{-1}(\beta)\, d\beta,\ A \in \mathcal{F}. \tag{55}$$

LEMMA 4.5. *Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing right continuous non-negative valued function on non-negative reals, and $F^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ its right continuous inverse function defined by*

$$F^{-1}(\beta) = \inf\{z \in \mathbb{R}_+ \mid F(z) > \beta\},\ \beta \in \mathbb{R}_+\,. \tag{56}$$

If $a \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}_+$ satisfy $F(a) = \alpha$ or $F^{-1}(\alpha) = a$ then

$$\int_0^\alpha F^{-1}(\beta)\, d\beta + \int_0^a F(z)\, dz = a\,\alpha \tag{57}$$

holds. If, in addition to $F(a) = \alpha$ or $F^{-1}(\alpha) = a$, $F(+\infty) := \lim_{z\to\infty} F(z) < +\infty$ and $\int_0^\infty (F(+\infty) - F(z))\, dz < +\infty$ hold, then

$$\int_\alpha^{F(+\infty)} F^{-1}(\beta)\, d\beta - \int_a^{+\infty} (F(+\infty) - F(z))\, dz = a\,(F(+\infty) - \alpha) \tag{58}$$

also holds.

PROOF. The formulas hold because both hand sides of each formula are 2-dimensional Lebesgue measure (area) of the same rectangle; both hand sides of (57) are area of the rectangle $[0, a] \times [0, \alpha]$, and both hand sides of (57) are area of the rectangle $[0, a] \times [\alpha, F(+\infty)]$. □

PROOF OF PROPOSITION 4.4.

Using (49) with $f = \frac{d\,\mu}{d\nu}$ and $a = y$ again, we have

$$\begin{aligned} \int_y^\infty (\nu(\Omega) - F_\mu(z))\, dz &= \int_y^\infty \nu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > z\})\, dz \\ &= \mu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y\}) - y\,(\nu(\Omega) - F_\mu(y)). \end{aligned} \tag{59}$$

In particular, $\displaystyle\int_0^\infty (\nu(\Omega) - F_\mu(z))\,dz \leqq \mu(\Omega) < \infty$.

Let $A \in \mathcal{F}$, and put

$$\begin{aligned} y_A &= \inf\{z \in \mathbb{R}_+ \mid \nu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > z\}) < \nu(A)\} \\ &= \inf\{z \in \mathbb{R}_+ \mid F_\mu(z) > \nu(A^c)\} = F_\mu^{-1}(\nu(A^c)). \end{aligned}$$

Then from (44) we have

$$\begin{aligned} v_\mu(A) &= \int_0^\infty (\nu(\Omega) - F_\mu(z)) \wedge \nu(A)\,dz = y_A\,\nu(A) + \int_{y_A}^\infty (\nu(\Omega) - F_\mu(z))\,dz. \\ &= \nu(A)\,F_\mu^{-1}(\nu(A^c)) + \int_{F_\mu^{-1}(\nu(A^c))}^\infty (\nu(\Omega) - F_\mu(z))\,dz. \end{aligned}$$

Substituting $\alpha = \nu(A^c)$ and $a = F_\mu^{-1}(\nu(A^c))$ in (58), we have (55). □

Proposition 4.6. *Let $(\Omega, \mathcal{F})$ be a measurable space satisfying $\mathcal{X} \neq \emptyset$, and $\nu \in \mathcal{M}(\Omega)$. For a finite measure $\mu \in \mathcal{M}(\Omega)$ satisfying $\mu \ll \nu$, denote the distribution function of the Radon–Nykodim derivative $\dfrac{d\,\mu}{d\nu}$ by $F_\mu : \mathbb{R}_+ \to \mathbb{R}_+$, as in (43), and define a set function $v_\mu : \mathcal{F} \to \mathbb{R}$ by (44).*

Then, for a non-negative valued measurable function $f : \Omega \to \mathbb{R}_+$, the Choquet integration $v(f)$ of f with respect to v satisfies

$$v_\mu(f) = \int_0^{\nu(\Omega)} F_\mu^{-1}(\beta)\,F_f^{-1}(\beta)\,d\beta, \tag{60}$$

where F_f is the distribution function of f defined by

$$F_f(z) = \nu(\{\omega \in \Omega \mid f(\omega) \leqq z\}),\ z \in \mathbb{R}_+, \tag{61}$$

and F_μ^{-1} and F_f^{-1} are respectively the right continuous inverse function of F_μ and F_f defined as in (54).

Proof. The definitions (21), (44), and $\nu(\{\omega \in \Omega \mid f(\omega) > y\}) = \nu(\Omega) - F_f(y)$ imply

$$\begin{aligned} v_\mu(f) &= \int_0^\infty v_\mu(\{\omega \in \Omega \mid f(\omega) > y\})\,dy \\ &= \int_{{\mathbb{R}_+}^2} (\nu(\Omega) - F_\mu(z)) \wedge \nu(\{\omega \in \Omega \mid f(\omega) > y\})\,dy\,dz \\ &= \int_{{\mathbb{R}_+}^2} (\nu(\Omega) - F_\mu(z)) \wedge (\nu(\Omega) - F_f(y))\,dy\,dz \\ &= \int_{F_f(y) > F_\mu(z)} (\nu(\Omega) - F_f(y))\,dy\,dz + \int_{F_\mu(z) \geqq F_f(y)} (\nu(\Omega) - F_\mu(z))\,dy\,dz. \end{aligned} \tag{62}$$

For the first term in the right hand side, we perform the y integration first. Note that since we chose F_f^{-1} to be right continuous,

$$y > F_f^{-1}(F_\mu(z)) \ \Rightarrow\ F_f(y) > F_\mu(z) \ \Rightarrow\ y \geqq F_f^{-1}(F_\mu(z))\ (\ \Rightarrow\ F_f(y) \geqq F_\mu(z)\ )$$

holds. Changing the integration variable from z to y in (58), choosing $F = F_f$, $\alpha = F_\mu(z)$, and $a = F_f^{-1}(F_\mu(z))$, and noting $F_\mu(+\infty) = F_f(+\infty) = \nu(\Omega)$ and recalling that a set of countable points has zero 1-dimensional Lebesgue measure, we have

$$\begin{aligned}&\int_{F_f(y)>F_\mu(z)} (\nu(\Omega) - F_f(y))\,dy = \int_{F_f^{-1}(F_\mu(z))}^{+\infty} (\nu(\Omega) - F_f(y))\,dy \\ &= \int_{F_\mu(z)}^{\nu(\Omega)} F_f^{-1}(\beta)\,d\beta - F_f^{-1}(F_\mu(z))\,(\nu(\Omega) - F_\mu(z)).\end{aligned}$$

Using this in (62) with Fubini's theorem and noting

$$F_\mu(z) < \beta \;\Rightarrow\; z < F^{-1}(\beta) \;\Rightarrow\; F_\mu(z) \leqq \beta \;(\Rightarrow\; z \leqq F^{-1}(\beta)\;),$$

we have

$$\begin{aligned}v_\mu(f) &= \int_{F_\mu(z)\leqq\beta\leqq\nu(\Omega)} F_f^{-1}(\beta)\,d\beta\,dz \\ &\quad - \int_0^\infty F_f^{-1}(F_\mu(z))\,(\nu(\Omega) - F_\mu(z))\,dz + \int_{F_\mu(z)\geqq F_f(y)} (\nu(\Omega) - F_\mu(z))\,dy\,dz \\ &= \int_0^{\nu(\Omega)} F_\mu^{-1}(\beta)\,F_f^{-1}(\beta)\,d\beta \\ &\quad - \int_0^\infty F_f^{-1}(F_\mu(z))\,(\nu(\Omega) - F_\mu(z))\,dz + \int_{F_f(y)\leqq F_\mu(z)} (\nu(\Omega) - F_\mu(z))\,dy\,dz.\end{aligned}$$

Noting that

$$(\; F_f(y) < F_\mu(z) \;\Rightarrow\;)\; y < F_f^{-1}(F_\mu(z)) \;\Rightarrow\; F_f(y) \leqq F_\mu(z) \Rightarrow\; y \leqq F_f^{-1}(F_\mu(z)),$$

we see that the last 2 terms cancel, which proves (60). □

The expression in the right hand side of (60) corresponds to the left hand side of the formula in the statement of [6, Lemma 11].

[6, Theorem 7] which corresponds to Theorem 4.3 uses yet another expression. Let m be the measure on a unit interval $(0, 1]$ defined as a Stieltjes measure satisfying

$$\begin{aligned}m((0,\gamma]) &= \frac{1}{\mu(\Omega)} \int_{\nu(\{\omega\in\Omega|\ \frac{d\,\mu}{d\nu}(\omega) > y\}) \leqq \gamma\nu(\Omega))} \nu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y\})\,dy \\ &= \frac{1}{\mu(\Omega)} \int_{F_\mu(y)\geqq(1-\gamma)\nu(\Omega)} (F_\mu(+\infty) - F_\mu(y))\,dy,\ \gamma \in (0, 1].\end{aligned} \tag{63}$$

Since F_μ^{-1} is defined to be right continuous, we have

$$\begin{aligned}m((0,\gamma)) &= \frac{1}{\mu(\Omega)} \int_{F_\mu(y)>(1-\gamma)\nu(\Omega)} (F_\mu(+\infty) - F_\mu(y))\,dy \\ &= \frac{1}{\mu(\Omega)} \int_{y\geq F_\mu^{-1}((1-\gamma)\nu(\Omega))} (F_\mu(+\infty) - F_\mu(y))\,dy \\ &= \frac{1}{\mu(\Omega)} \int_{F_\mu^{-1}((1-\gamma)\nu(\Omega))}^{\infty} (F_\mu(+\infty) - F_\mu(y))\,dy\end{aligned} \tag{64}$$

for an open interval $(0, \gamma)$.

Fubini's Theorem implies,

$$m((0,1]) = \frac{1}{\mu(\Omega)} \int_{\mathbb{R}_+} \nu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y\})\, dy = \frac{1}{\mu(\Omega)} \int_{\frac{d\,\mu}{d\nu}(\omega) > y} \nu(d\omega)\, dy$$
$$= \frac{1}{\mu(\Omega)} \int_\Omega \left(\int_0^{\frac{d\,\mu}{d\nu}(\omega)} dy \right) \nu(d\omega) = \frac{1}{\mu(\Omega)} \int_\Omega \frac{d\,\mu}{d\nu}(\omega)\, \nu(d\omega) = \frac{1}{\mu(\Omega)} \int_\Omega \mu(d\omega) = 1,$$

hence, m is a probabiity measure on $(0, 1]$.

LEMMA 4.7. *It follows that*

$$g(\gamma) := \int_{[1-\gamma,1]} \frac{m(d\alpha)}{\alpha} = \frac{\nu(\Omega)}{\mu(\Omega)} F_\mu^{-1}(\nu(\Omega)\gamma),\ \gamma \in [0,1). \tag{65}$$

PROOF. Not that for $\gamma \in [0, 1)$

$$\{(\alpha,\beta) \in (0,1] \times [0,1) \mid 1-\gamma \leqq \beta < 1,\ 1-\beta \leqq \alpha \leqq 1\}$$
$$= \{(\alpha,\beta) \in (0,1] \times [0,1) \mid 0 < \alpha \leqq 1,\ 1-(\alpha \wedge \gamma) \leqq \beta < 1\},$$

hence $\displaystyle\int_{[1-\gamma,1)} g(\beta)\, d\beta = \int_{(0,1]} \frac{\alpha \wedge \gamma}{\alpha}\, m(d\alpha) = m((0,\gamma)) + \gamma\, g(1-\gamma)\,.$

On the other hand, substituting $F = F_\mu$, $\alpha = (1-\gamma)\nu(\Omega)$, and $a = F_\mu^{-1}(\alpha)$ in (58), and then using (64). and changing integration variables as $\beta = \beta'\nu(\Omega)$, and also using $F_\mu(+\infty) = \nu(\Omega)$, we have

$$\frac{\nu(\Omega)}{\mu(\Omega)} \int_{1-\gamma}^1 F_\mu^{-1}(\beta'\,\nu(\Omega))\, d\beta' - m((0,\gamma)) = \frac{\nu(\Omega)}{\mu(\Omega)}\, \gamma\, F_\mu^{-1}((1-\gamma)\nu(\Omega)),\ 0 \leqq \gamma \leqq 1.$$

Therefore, if we define $h:\ (0,1] \to \mathbb{R}$ by $h(\gamma) = \dfrac{\nu(\Omega)}{\mu(\Omega)} F_\mu^{-1}((1-\gamma)\nu(\Omega)) - g(1-\gamma)$, we have $\dfrac{1}{x} \displaystyle\int_{(0,x]} h(\beta)\, d\beta = h(x),\ x \in (0,1]$, which implies that h is a constant. Substituting $\gamma = 1$ in (63) and (64), we see that

$$g(0) = m(\{1\}) = \frac{1}{\mu(\Omega)} \int_{F_\mu(y)=0} (F_\mu(+\infty) - F_\mu(y))\, dy = \frac{\nu(\Omega)}{\mu(\Omega)} F_\mu^{-1}(0)$$

which implies $h(1) = 0$. Therefore (65) holds. □

PROPOSITION 4.8. *Let $(\Omega, \mathcal{F})$ be a measurable space satisfying $\mathcal{X} \neq \emptyset$, and $\nu \in \mathcal{M}(\Omega)$. Let $\mu \in \mathcal{M}(\Omega)$ be a finite measure satisfying $\mu \ll \nu$, and $v_\mu :\ \mathcal{F} \to \mathbb{R}$ be the submodular function (44), and m be the measure on $(0,1]$ defined in (63). Then, for a non-negative*

valued measurable function $f:\ \Omega \to \mathbb{R}_+$, the Choquet integration $v_\mu(f)$ of f with respect to v_μ satisfies,

$$
\begin{aligned}
&v_\mu(f) = \int_{(0,1]} v_\alpha(f)\, m(d\alpha), \ \textit{where,}\\
&v_\alpha(f) = \frac{\mu(\Omega)}{\alpha\nu(\Omega)} \int_0^\infty \nu(\{\omega \in \Omega \mid f(\omega) > z\}) \wedge (\alpha\nu(\Omega))\, dz,\ 0 < \alpha \leqq 1.
\end{aligned} \tag{66}
$$

Proof. The definition (44) of v_μ and the definition (21) of Choquet integration imply, as in the proof of (46) in Theorem 4.2,

$$
v_\mu(f) = \int_0^\infty \int_0^\infty (\nu(\Omega) - F_\mu(y)) \wedge \nu(\{\omega \in \Omega \mid f(\omega) > z\})\, dy\, dz. \tag{67}
$$

On the other hand, using (64) with $\gamma = 1 - \dfrac{F_f(z)}{\nu(\Omega)}$ and (65) with $\gamma = \dfrac{F_f(z)}{\nu(\Omega)}$ we have

$$
\begin{aligned}
&\int_{(0,1]} \nu(\{\omega \in \Omega \mid f(\omega) > z\}) \wedge (\alpha\nu(\Omega)) \frac{m(d\alpha)}{\alpha\nu(\Omega)}\\
&= m((0, 1 - \frac{F_f(z)}{\nu(\Omega)})) + (1 - \frac{F_f(z)}{\nu(\Omega)}) \int_{[1-\frac{F_f(z)}{\nu(\Omega)},1]} \frac{m(d\alpha)}{\alpha}\\
&= \frac{1}{\mu(\Omega)} \int_{F_\mu^{-1}(F_f(z))}^\infty (\nu(\Omega) - F_\mu(y))\, dy + \frac{1}{\mu(\Omega)}\, F_\mu^{-1}(F_f(z))\, (\nu(\Omega) - F_f(z)).
\end{aligned}
$$

Using the right continuity of F_μ^{-1} and the fact that a point has zero 1-dimensional Lebesgue measure in a similar way as in the proof of Proposition 4.6, we see that the right hand side is equal to $\dfrac{1}{\mu(\Omega)} \displaystyle\int_0^\infty (\nu(\Omega) - F_\mu(y)) \wedge \nu(\{\omega \in \Omega \mid f(\omega) > z\})\, dy$. Therefore

$$
\begin{aligned}
&\int_{(0,1]} \nu(\{\omega \in \Omega \mid f(\omega) > z\}) \wedge (\alpha\nu(\Omega)) \frac{m(d\alpha)}{\alpha\nu(\Omega)}\\
&= \frac{1}{\mu(\Omega)} \int_0^\infty (\nu(\Omega) - F_\mu(y)) \wedge \nu(\{\omega \in \Omega \mid f(\omega) > z\})\, dy.
\end{aligned}
$$

Integrating over $z \geqq 0$ and using (67), we have (66). □

We gave several formulas expressing values of submodular functions for sets or Choquet integrations of functions in terms of supremum over class of measures, such as (9) and (2) of Theorem 2.7 in § 2, (36) of Theorem 3.2 in § 3, and (51) of Theorem 4.3 in § 4. Note that formulas in § 4 for ν-uniform ('law invariant') submodular functions and those in § 2 and § 3 have different classes of measures to take supremum. How far they are to be mathemtaically united is not clear to the author.

5 Recursion relation associated with the fundamental formula

Let us return to the recursion relation (10) in § 1 for set functions on a measurable space $(\Omega, \mathcal{F})$ satisfying $\mathcal{X} \neq \emptyset$. Definition 2.1 of continuity of set function in this paper implies that if v_n is continuous, then the measures $\mu_{v_n,\mathcal{I}}$, $\mathcal{I} \in \mathcal{X}$, in the right hand side of (10) are continuous (σ-additive) and well-defined as measures, hence (10) makes sense.

We do not know in general if the resulting set function v_{n+1} is continous. On the other hand, if v_n , $n = 0, 1, 2, \ldots$, are all continuous for some $(\Omega, \mathcal{F})$ and v_0 , then the recursion converges.

PROPOSITION 5.1. *Let $(\Omega, \mathcal{F})$ be a meassurable space satisfying $\mathcal{X} \neq \emptyset$, and $v_0 : \mathcal{F} \to \mathbb{R}_+$ a non-decreasing and continuous set function satisfying $v_0(\emptyset) = 0$. If $v_n : \mathcal{F} \to \mathbb{R}_+$, $n = 0, 1, 2, \ldots$, is a sequence of continuous set functions which satisfies (10), then for each $A \in \mathcal{F}$, the sequence $v_n(A)$, $n = 0, 1, 2, \ldots$, in non-decreasing and bounded from above by $v_0(\Omega)$, hence converges. The limit $v : \mathcal{F} \to \mathbb{R}_+$ defined by $v(A) = \lim_{n\to\infty} v_n(A)$, $A \in \mathcal{F}$, is non-decreasing, submodular, and satisfies (14) and $v(\emptyset) = 0$ and $v(\Omega) = v_0(\Omega)$.*

PROOF. Since we assume continuity of v_n for all $n \in \mathbb{Z}_+$, $\mu_{v_n,\mathcal{I}}$ are measures. The recursion relation therefore implies that v_{n+1} is non-decreasing and $v_{n+1}(\emptyset) = 0$ and $v_{n+1}(\Omega) = v_0(\Omega)$ for all $n \in \mathbb{Z}_+$, hence $v(\emptyset) = \lim_{n\to\infty} v_n(\emptyset) = 0$ and $v(\Omega) = \lim_{n\to\infty} v_n(\Omega) = v_0(\Omega)$. Since v_{n+1} is non-decrasing, $v_{n+1}(A) \leqq v_{n+1}(\Omega) = v_0(\Omega)$ for all n and A. For $A \in \mathcal{F}$, Note that $\mathcal{I}_A \in \mathcal{X}$, the insertion of A to $\mathcal{I} \in \mathcal{X}$ defined by (7) satisfies $A \in \mathcal{I}_A$, which, with (8), implies

$$v_{n+1}(A) = \sup_{\mathcal{I}\in\mathcal{X}} \mu_{v_n,\mathcal{I}}(A) \geqq \mu_{v_n,\mathcal{I}_A}(A) = v_n(A),$$

hence the sequence $v_n(A)$, $n = 0, 1, 2, \ldots$, is non-decreasing. We have seen that the sequence is bounded from above, hence it converges. Thus

$$v(A) = \lim_{n\to\infty} v_n(A) = \sup_{n\in\mathbb{N}} v_n(A), \ A \in \mathcal{F}. \tag{68}$$

The non-decreasing property of measures and (10) imply that the limit v is also non-decreasing.

To prove that v is submodular, let $A, B \in \mathcal{F}$. Then $v(A \cap B) = \lim_{n\to\infty} v_n(A \cap B)$ and $v(A \cup B) = \lim_{n\to\infty} v_n(A \cup B)$ imply that for any $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$v(A \cap B) + v(A \cup B) \leqq v_n(A \cap B) + v_n(A \cup B) + \epsilon. \tag{69}$$

Let $(\mathcal{I}_A)_B \in \mathcal{X}$ be the sequential insertion of A and B to $\mathcal{I}$ in (16). Then as remarked below (16), $A \cap B \in (\mathcal{I}_A)_B$ and $A \cup B \in (\mathcal{I}_A)_B$, hence by definition of extremal measure

in (8), additivity of measure, (9), and (68) it holds that

$$\begin{aligned} v_n(A\cap B)+v_n(A\cup B) &= \mu_{v_n,(\mathcal{I}_A)_B}(A\cap B)+\mu_{v_n,(\mathcal{I}_A)_B}(A\cup B)\\ &= \mu_{v_n,(\mathcal{I}_A)_B}(A)+\mu_{v_n,(\mathcal{I}_A)_B}(B)\\ &\leqq v_{n+1}(A)+v_{n+1}(B)\\ &\leqq v(A)+v(B). \end{aligned}$$

Since $\epsilon>0$ is arbitrary, this and (69) proves the submodularity $v(A\cap B)+v(A\cup B)\leqq v(A)+v(B)$.

To prove (14) asuume that $\mathcal{A}=\{A_n \mid n=1,2,3,\ldots\}\subset\mathcal{F}$ satisfies $A_1\subset A_2\subset\cdots$. Since by assumption v_k is continuous, and since we saw that v_k is non-decreasing, Proposition 2.3 implies (14) for v_k: $\lim_{n\to\infty} v_k(A_n)=v_k(\bigcup_{n\in\mathbb{N}} A_n),\ k\in\mathbb{Z}_+$. This and $v(\bigcup_{n\in\mathbb{N}} A_n)=\lim_{k\to\infty} v_k(\bigcup_{n\in\mathbb{N}} A_n)$ imply that for any $\epsilon>0$ there exists $k\in\mathbb{Z}_+$ such that

$$v(\bigcup_{n\in\mathbb{N}} A_n)\leqq v_k(\bigcup_{n\in\mathbb{N}} A_n)+\epsilon=\lim_{n\to\infty} v_k(A_n)+\epsilon.$$

Since v is non-decreasing and $v(A_n)=\sup_{k\in\mathbb{N}} v_k(A_n)$ we further have

$$\lim_{n\to\infty} v(A_n)\leqq v(\bigcup_{n\in\mathbb{N}} A_n)\leqq \lim_{n\to\infty} v(A_n)+\epsilon.$$

Since $\epsilon>0$ is arbitraly this proves (14). □

Note that we assume continuity of each v_n in this result. The continuity of the limit v, or equivalently, (15) for v, is an open problem.

If the total set is a finite set, then there are only finite number of distinct subsets, hence if a set function is finitely additive then it is σ-additive (continuous). In other words, any finitely additive measures is a (σ-additive) measure if the total set is a finite set. Therefore, Definition 2.1 of continuity of set function is always satisfied, and in particular, the recursion relation (10) makes sense for any non-decreasing initial set function v_0 and all $n\in\mathbb{N}$.

Hereafter, we assume that the cardinality m of the total set $\Omega=\Omega_m$ is finite: $m\in\mathbb{N}$, and assume for simplicity $\mathcal{F}=2^{\Omega_m}$, and consider the recursion relation (10): $v_{n+1}=\sup_{\mathcal{I}\in\mathcal{X}}\mu_{v_n,\mathcal{I}}$, $n=0,1,2,\ldots$. with $v_0:\ 2^{\Omega_m}\to\mathbb{R}_+$ a non-decreasing (automatically continuous) set function satisfying $v_0(\emptyset)=0$. Proposition 5.1 implies that $v_n(A)$, $n\in\mathbb{Z}_+$, is non-decreasing in n and converges as $n\to\infty$, for each $A\in\mathcal{F}$.

Note that if we put $\Omega_m=\{\omega_i \mid i=1,\ldots,m\}$, then $\mathcal{I}\in\mathcal{X}$ is of a form

$$\mathcal{I}=\{\emptyset,\{\omega_{i_1}\},\{\omega_{i_1},\omega_{i_2}\},\ldots,\{\omega_{i_1},\ldots,\omega_{i_{m-1}}\},\Omega_m\}. \tag{70}$$

In particular, the cardinality $\sharp\mathcal{X}$ of $\mathcal{X}$ satisfies $\sharp\mathcal{X} \leqq m!$. There are at most $m!$ numbers to compare in the supremum in the right hand side of (10), hence supremum is always attained; for each $A \subset \Omega_m$ there exists $\mathcal{I} \in \mathcal{X}$ such that

$$v_{n+1}(A) = \mu_{v_n,\mathcal{I}}(A). \tag{71}$$

If the limit is attained after a finite iteration of (10), namely, if $v_{n+1} = v_n$ for an integer n, then (10) implies $v_n = \sup_{\mathcal{I}\in\mathcal{X}} \mu_{v_n,\mathcal{I}}$, which implies (9) for $v = v_n$ and Theorem 2.7 then implies that v_n is submodular. On the other hand, if we start from v_0 which is not submodular, then Theorem 2.7 implies $v_1 \neq v_0$. Thus the recursion relation (10) is of interest in relating non-submodular functions to submodular functions.

As a first elementary example, we will prove that if $m = 3$, then for any non-decreasing v_0 we have $v_2 = v_1$, so that the limit is reached in at most 1 step recursion and v_1 is submodular.

PROPOSITION 5.2. *Let $m \in \mathbb{N}$ and Ω_m be a set of cardinality $\sharp\Omega_m = m$.*

(i) For any set function $v : 2^{\Omega_m} \to \mathbb{R}_+$ if one of the following (i)–(vii) holds for a pair of sets $A, B \subset \Omega_m$, then $v(A \cap B) + v(A \cup B) = v(A) + v(B)$ holds; (i) $A = \emptyset$, or (ii) $B = \emptyset$, or (iii) $A = \Omega_m$, or (iv) $B = \Omega_m$, or (v) $A \subset B$, or (vi) $A \supset B$, or (vii) $A = B$.

(ii) Let $v_0 : 2^{\Omega_m} \to \mathbb{R}_+$ be a non-decreasing set function satisfying $v_0(\emptyset) = 0$, and for $n \in \mathbb{N}$, let $v_n : 2^{\Omega_m} \to \mathbb{R}_+$ be the non-decreasing set function determined by the recursion relation (10). Then for any positive integer $n \in \mathbb{N}$, if one of the following (i)–(iv) holds for a pair of sets $A, B \subset \Omega_m$, then $v_n(A \cap B) + v_n(A \cup B) \leqq v_n(A) + v_n(B)$ holds; (i) $A \cap B = \emptyset$, (ii) $A \cup B = \Omega_m$, (iii) $\sharp A = 1$, (iv) $\sharp A = m - 1$, where $\sharp A$ denotes the cardinality of (number of elements in) the set A.

PROOF. (i) By interchanging A and B, it suffices to prove (i), (iv), (v). If $A = \emptyset$ then $v(A \cap B) + v(A \cup B) = v(\emptyset) + v(B) = v(A) + v(B)$, If $B = \Omega_m$ then $v(A \cap B) + v(A \cup B) = v(A) + v(\Omega_m) = v(A) + v(B)$, If $A \subset B$ then $v(A \cap B) + v(A \cup B) = v(A) + v(B)$.

(ii) (i) If $A \cap B = \emptyset$, then as noted at (71), there exists $\mathcal{I} \in \mathcal{X}$ such that $v_n(A \cup B) = \mu_{v_{n-1},\mathcal{I}}(A \cup B)$. Using the additivity of the measure $\mu_{v_{n-1},\mathcal{I}}$ with the assumption $A \cap B = \emptyset$, we have

$$\begin{aligned} &v_n(A \cap B) + v_n(A \cup B) = \mu_{v_{n-1},\mathcal{I}}(A \cup B) = \mu_{v_{n-1},\mathcal{I}}(A) + \mu_{v_{n-1},\mathcal{I}}(B) \\ &\leqq v_n(A) + v_n(B). \end{aligned}$$

(ii) If $A \cup B = \Omega_m$, then as in the case (i) there exists $\mathcal{I} \in \mathcal{X}$ such that $v_n(A \cap B) = \mu_{v_{n-1},\mathcal{I}}(A \cap B)$. Noting that $v_n(\Omega_m) = v_0(\Omega_m) = \mu_{v_{n-1},\mathcal{I}}(\Omega_m)$ for all $\mathcal{I} \in \mathcal{X}$, and using the aditivity of the measure $\mu_{v_{n-1},\mathcal{I}}$, we have

$$\begin{aligned} &v_n(A \cap B) + v_n(A \cup B) = \mu_{v_{n-1},\mathcal{I}}(A \cap B) + \mu_{v_{n-1},\mathcal{I}}(\Omega_m) \\ &= \mu_{v_{n-1},\mathcal{I}}(A) + \mu_{v_{n-1},\mathcal{I}}(B) \leqq v_n(A) + v_n(B). \end{aligned}$$

(iii) If $\sharp A = 1$ then, either $A \cap B = \emptyset$ or $A \subset B$ holds for any $B \subset \Omega_m$. In both cases we already proved that $v_n(A \cap B) + v_n(A \cup B) \leqq v_n(A) + v_n(B)$ holds.

(iv) If $\sharp A = m - 1$ then, either $A \cup B = \Omega_m$ or $A \supset B$ holds for any $B \subset \Omega_m$. In both cases we already proved that $v_n(A \cap B) + v_n(A \cup B) \leqq v_n(A) + v_n(B)$ holds. □

Corollary 5.3. *If $m = \sharp\Omega_m = 3$ and $v_0 : 2^{\Omega_3} \to \mathbb{R}_+$, is non-decreasing and $v_0(\emptyset) = 0$, then the sequence $v_n : 2^{\Omega_3} \to \mathbb{R}_+$, $n \in \mathbb{N}$, determined by the recursion relation (10) satisfies $v_1 = v_2 = \cdots$ and v_1 is a submodular function.*

Proof. If $A \subset \Omega_3$, then $0 \leqq \sharp A \leqq \sharp\Omega_3 = 3$. This implies $A = \emptyset$ or $A = \Omega_3$ or $\sharp A = 1$ or $\sharp A = 2 = m - 1$. Proposition 5.2 implies that for all the cases, $v_n(A \cap B) + v_n(A \cup B) \leqq v_n(A) + v_n(B)$ holds for any $B \subset \Omega_3$ and for all $n = 1, 2, \ldots$. Therefore v_1 is submodular and the recursion relation reaches the limit at $n = 1$. □

It turns out that not only for $m = 3$, but for all $m \in \mathbb{N}$ we can easily find examples that the recursion relation (10) reaches the limit after just 1 iteration.

Proposition 5.4. *Let $m = \sharp\Omega_m \in \mathbb{N}$ and $v_0 : 2^{\Omega_m} \to \mathbb{R}_+$ be non-decreasing and $v_0(\emptyset) = 0$, and assume that there exists $g : \{1, 2, \ldots, m\} \to \mathbb{R}_+$ such that $v_0(A) = g(\sharp A)$, $A \subset \Omega_m$. Then the recursion (10) reaches the limit at $n = 1$, i.e., $v_2 = v_1$, and hence v_1 is submodular.*

Proof. Let S_m denote the collection of the rearrangements of the sequence of m elements of the total set Ω_m;

$$S_m = \{(\sigma_1, \ldots, \sigma_m) \mid \{\sigma_1, \ldots, \sigma_m\} = \Omega_m\}, \tag{72}$$

and for $\sigma = (\sigma_1, \ldots, \sigma_m) \in S_m$ put

$$\mathcal{I}_\sigma = \{\emptyset, \{\sigma_1\}, \{\sigma_1, \sigma_2\}, \ldots, \{\sigma_1, \ldots, \sigma_{m-1}\}, \Omega_m\}. \tag{73}$$

As discussed at around (70) we have

$$\mathcal{X} = \{\mathcal{I}_\sigma \mid \sigma \in S_m\}. \tag{74}$$

The definition (8) of extremal measure implies, with the assumption $v_0(A) = g(\sharp A)$,

$$(75)\quad \mu_{v_0,\mathcal{I}_\sigma}(\{\sigma_i\}) = v_0(\{\sigma_1,\ldots,\sigma_i\}) - v_0(\{\sigma_1,\ldots,\sigma_{i-1}\}) = g(i) - g(i-1),\ \ i = 1,\ldots,m.$$

Note that $g(0) = v_0(\emptyset) = 0$. Since v_0 is non-decreasing, so is g. In particular, $g(i) - g(i-1) \geqq 0$, $i = 1,\ldots,m$.

Choose the rearrangement $\alpha = (\alpha_1,\ldots,\alpha_m)$ of first m positive integers $\{1,\ldots,m\}$ such that

$$(76)\qquad g(\alpha_1) - g(\alpha_1 - 1) \geqq g(\alpha_2) - g(\alpha_2 - 1) \geqq \cdots \geqq g(\alpha_m) - g(\alpha_m - 1) \geqq 0.$$

In particular, (75) and (76) imply $g(\alpha_j) - g(\alpha_j - 1) = \mu_{v_0,\mathcal{I}_\sigma}(\{\sigma_{\alpha_j}\})$, $\sigma \in S_m$.

Put $\Omega_m = \{\omega_1,\ldots,\omega_m\}$. Since $\{\sigma_{\alpha_j} \mid j = 1,\ldots,m\} = \Omega_m$, there exists $\sigma \in S_m$ dependent rearrangement of first m positive integers $h_\sigma :\ \{1,\ldots,m\} \to \{1,\ldots,m\}$ such that $\sigma_{\alpha_{h_\sigma(i)}} = \omega_i$, $i = 1,\ldots,m$. Then

$$(77)\qquad \mu_{v_0,\mathcal{I}_\sigma}(\{\omega_i\}) = g(\alpha_{h_\sigma(i)}) - g(\alpha_{h_\sigma(i)} - 1),\ i = 1,\ldots,m.$$

Hence, for $A \in 2^{\Omega_m}$,

$$v_1(A) = \sup_{\sigma\in S_m} \mu_{v_0,\mathcal{I}_\sigma}(A) = \sup_{\sigma\in S_m} \sum_{\omega_i\in A} \mu_{v_0,\mathcal{I}_\sigma}(\{\omega_i\}) = \sup_{\sigma\in S_m} \sum_{\omega_i\in A} (g(\alpha_{h_\sigma(i)}) - g(\alpha_{h_\sigma(i)} - 1)).$$

Supremum is attained by choosing largest $\sharp A$ terms in (76), hence

$$(78)\qquad v_1(A) = \sum_{j=1}^{\sharp A} (g(\alpha_j) - g(\alpha_j - 1)).$$

We can repeat the calculations by substituting v_0 for v_1, to find

$$\mu_{v_1,\mathcal{I}_\sigma}(\{\omega_i\}) = g(\alpha_{h_\sigma(i)}) - g(\alpha_{h_\sigma(i)} - 1),\ i = 1,\ldots,m,$$

whose right hand side is equal to (77), hence

$$v_2(A) = \sup_{\sigma\in S_m} \mu_{v_1,\mathcal{I}_\sigma}(A) = \sum_{j=1}^{\sharp A} (g(\alpha_j) - g(\alpha_j - 1)) = v_1(A),\ A \in \mathcal{F},$$

which implies $v_1 = \sup_{\mathcal{I}\in\mathcal{X}} \mu_{v_1,\mathcal{I}}$. □

With all these results for examples of $v_2 = v_1$, it may be interesting to see that there is an example of $v_2 \neq v_1$ for $m = 4$. The table below gives such an example. For example, we see from the table that $v_1(\{1,2\}) = 17 \neq 18 = v_2(\{1,2\})$, hence v_1 is not submodular and it is not the limit of the recursion. We can also see

$$v_1(\{1\}) + v_1(\{1,2,3\}) = 11 + 31 = 42 > 41 = 17 + 24 = v_1(\{1,2\}) + v_1(\{1,3\})$$

which directly proves that v_1 is not submodular.

A	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	Ω_4
$v_0(A)$	0	7	13	20	19	17	24	30	28	34	41	31	36	42	43	44
$v_1(A)$	0	11	15	22	23	17	24	30	28	34	41	31	36	42	43	44
$v_2(A)$	0	11	15	22	23	18	25	30	29	34	41	31	36	42	43	44

Incidentally, in this example we have $v_3 = v_2$, so that the recursion (10) reaches the limit at $n = 2$, and v_2 is submodular. It is an open problem whether there exists an infinite sequence v_n , $n = 0, 1, 2, \ldots$, of non-decreasing and non-submodular sset functions which satisfies the recursion (10).

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