Recursion relation for set functions on infinite set associated to representation of submodular function as supremum of measures

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Abstract

Based on the author's previous study on the set functions and Choquet integrations, on measurable spaces having chain of measurable sets which generates the sigma-algebra, we continue and extend the study on a recursion relation of set functions associated to the fundamental representation formula of submodular set functions in terms of supremum of the extremal measures. We prove, on a countably infinite total set, a result corresponding to the previous result on finite total sets, that a non-decreasing initial set function whose dependence on the set variable is determined by the cardinal of the set reaches the limit at first iteration. We also advance our study on the recursion on finite total sets, and complete our previous study on basic formula in mathematical finance on law invariant coherent risk measures.

1 Introduction

Let $(\Omega, \mathcal{F})$ be a measurable space, namely, $\mathcal{F}$ is a σ-algebra, a class of subsets of Ω closed under complements and countable unions. We first recall two fundamental objects $\mathcal{X}$ and $\mu_{v,\mathcal{I}}$ introduced in [5, 6]. $\mathcal{X}$ denotes the collection of the chains $\mathcal{I} \subset \mathcal{F}$ which contains the emptyset $\emptyset$ and the total set Ω, and generates $\mathcal{F}$. In other words, $\mathcal{X} = \{\mathcal{I} \subset \mathcal{F} \mid$ the following (1) holds$\}$, where

$$(1) \qquad \begin{cases} i)\ \ \sigma[\mathcal{I}] = \mathcal{F}, \text{ where } \sigma[\mathcal{I}] \text{ denotes the smallest } \sigma\text{-algebra containing } \mathcal{I}, \\ ii)\ \ \emptyset \in \mathcal{I},\ \Omega \in \mathcal{I}, \\ iii)\ \ \mathcal{I} \text{ is totally ordered with respect to inclusion,} \\ \quad \text{i.e., for all } I_1, I_2 \in \mathcal{I} \text{ either } I_1 \subset I_2 \text{ or } I_2 \subset I_1\,. \end{cases}$$

It is proved in [5, Proposition 7] that Polish spaces with Borel σ-algebra are examples of measurable spaces satisfying $\mathcal{X} \neq \emptyset$. A finite set $\Omega = \Omega_m = \{1, 2, \ldots, m\}$ with m elements and a countable set $\Omega = \mathbb{N} = \{1, 2, \ldots\}$ with all the subsets being measurable ($\mathcal{F} = 2^\Omega$) are simple examples of Polish spaces, as well as a wide class of spaces extensively used in the

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theory of stochastic processes. See [5] for further considerations including open problems concerning $\mathcal{X} \neq \emptyset$. We consider the measurable space $(\Omega, \mathcal{F})$ satisfying $\mathcal{X} \neq \emptyset$ throughout this paper.

Let $v : \mathcal{F} \to \mathbb{R}$ be a real valued non-decreasing set function satisfying $v(\emptyset) = 0$. Here, by non-decreasing we mean

$$v(A) \leqq v(B), \ A \subset B, \ A, B \in \mathcal{F}. \tag{2}$$

To recall the definition in [6, §2] of the extremal measure $\mu_{v,\mathcal{I}}$ of v corresponding to $\mathcal{I} \in \mathcal{X}$, first we define $\mu_{v,\mathcal{I}}$ on $\mathcal{I}$ by

$$\mu_{v,\mathcal{I}}(I) = v(I), \ I \in \mathcal{I}. \tag{3}$$

Denote by $\mathcal{J}$ the finitely additive class generated by $\mathcal{I}$, the smallest class of sets such that $\mathcal{I}$ is a subset and closed under complement and union. As in a standard elementary measure theory, $\mu_{v,\mathcal{I}}$ is uniquely extended to $\mathcal{J}$ as a finitely additive measure, for which we use the same notation and put $\mu_{v,\mathcal{I}} : \mathcal{J} \to \mathbb{R}_+$. We say that a non-decreasing set function $v : \mathcal{F} \to \mathbb{R}$ is continuous, if for every $\mathcal{I} \in \mathcal{X}$ the finitely additive measure $\mu_{v,\mathcal{I}} : \mathcal{J} \to \mathbb{R}_+$ is σ-additive, or equivalently, continuous, on the finitely additive class $\mathcal{J}$ [6, Definition 2.1]. A standard extension theorem of measures and $\sigma[\mathcal{I}] = \mathcal{F}$ in (1) then imply that $\mu_{v,\mathcal{I}}$ is uniquely extended to a measure on $\mathcal{F}$ for all $\mathcal{I} \in \mathcal{X}$ if v is continuous. We will use the same notation and call the measure $\mu_{v,\mathcal{I}} : \mathcal{F} \to \mathbb{R}$ the extremal measure of v corresponding to $\mathcal{I} \in \mathcal{X}$. If v is a measure satisfying $v(\Omega) \in \mathbb{R}_+$ (i.e., a finite measure), then $\mu_{v,\mathcal{I}} = v$, $\mathcal{I} \in \mathcal{X}$, hence the definition of the continuity of [6, Definition 2.1] is a generalization of a standard definition of the continuity of measures to general non-decreasing real valued set functions.

A set function $v : \mathcal{F} \to \mathbb{R}$ which satisfies

$$v(A) + v(B) \geqq v(A \cup B) + v(A \cap B), \quad A, B \in \mathcal{F}, \tag{4}$$

is called a submodular function. Extremal measures characterizes the submodular functions.

Theorem 1 ([6, Theorem 2.7]) *Let $(\Omega, \mathcal{F})$ be a measurable space satisfying $\mathcal{X} \neq \emptyset$. Let $v : \mathcal{F} \to \mathbb{R}$ be a non-decreasing and continuous set function, satisfying $v(\emptyset) = 0$. Then v is submodular, i.e., (4) holds, if and only if*

$$v = \sup_{\mathcal{I} \in \mathcal{X}} \mu_{v,\mathcal{I}} \tag{5}$$

holds. ◇

Theorem 1 is a generalization of the theory of cores for a finite set in [9] to measurable spaces satisfying $\mathcal{X} \neq \emptyset$. See [6] for a proof of Theorem 1 and other characterization of submodular functions in our framework. Motivated by (5) in Theorem 1, we started in [6] a study on the recursion relation

$$v_{n+1} = \sup_{\mathcal{I} \in \mathcal{X}} \mu_{v_n,\mathcal{I}}, \ n = 0, 1, 2, \ldots, \tag{6}$$

with $v_0 : \mathcal{F} \to \mathbb{R}$ being a non-decrasing continuous set function. If v_n is continuous, then by our definition of continuity, the measures $\mu_{v_n,\mathcal{I}}$, $\mathcal{I} \in \mathcal{X}$, in the right hand side of (6) are continuous (σ-additive), hence well-defined as measures on $\mathcal{F}$, so that (6) makes sense as an

equation of set function on $\mathcal{F}$. Theorem 1 then implies $v_1 \neq v_0$ if v_0 is not submodular and $v_1 = v_0$ if v_0 is submodular.

In this paper we prove the following. We use ∞ for the cardinality of a countable set, such as $\sharp\mathbb{N} = \infty$ for $\mathbb{N} = \{1, 2, \ldots\}$.

Theorem 2 *Let $v_0 : 2^{\mathrm{N}} \to \mathbb{R}_+$ be a non-decreasing set function on the measurable space $(\Omega, \mathcal{F}) = (\mathbb{N}, 2^{\mathrm{N}})$ satisfying $v_0(\emptyset) = 0$, and assume that there exists $g : \mathbb{N} \cup \{0, \infty\} \to \mathbb{R}_+$ such that $v_0(A) = g(\sharp A)$, $A \subset \mathbb{N}$. Then v_0 is continuous if and only if*

$$\lim_{n\to\infty} g(n) = g(\infty) \tag{7}$$

holds, and when (7) holds, the recursion (6) reaches the limit at $n = 1$, i.e., $v_2 = v_1$, and hence v_1 is submodular. ◇

A corresponding result for a finite total set ($(\Omega, \mathcal{F}) = (\Omega_m, 2^{\Omega_m})$ with $m \in \mathbb{N}$ and $\sharp\Omega_m = m$) is proved in [6, Proposition 5.4]. See [6, §5] also for other results concerning recursion relation (6). Note that for a finite total set, σ-additivity of a measure is equivalent to finite additivity, hence all the non-decreasing set functions are by definition continuous. In contrast, (7) indicates that our definition of continuity is non-trivial on infinite total sets.

Theorem 1 implies that the recursion (6) is of potential interest in relating general set functions to submodular functions. See [3, 4, 8, 9] and the references in [5, 6] for a part of large background and long history of the theory of submodular functions. In application to mathematical finance or cooparative game theory, relation between submodular functions and general set functions would be of interest in modelling risk measures and gains in cooperations when submodularity (or directly related property of convex game) is absent.

Theorem 2 concludes that with the assumptions on initial function v_0 in the theorem the recursion reaches the limit with 1 iteration, which is a result shared with the case of finite total set with similar assumptions in [6, Proposition 5.4]. One might anticipate that since the assumption $v_0(A) = g(\sharp A)$ implies $v_0(A) = v_0(\mathbb{N})$ for any infinite set $A \subset \mathbb{N}$, the assumption is too restrictive for infinite sets so that the conclusion for $\mathbb{N}$ did not change from those for the finite total sets. Concerning this point we note that there are examples for which there is an infinite set A with $v_0(A) < v_0(\mathbb{N})$ and $v_2 = v_1$ can be proved in an analogous way as the proof of Theorem 2. For example, the following holds.

Corollary 3 *Put $2\mathbb{N} - 1 = \{1, 3, 5, \ldots\}$ and $2\mathbb{N} = \{2, 4, 6, \ldots\}$. Let $v_0 : 2^{\mathrm{N}} \to \mathbb{R}_+$ be non-decreasing and $v_0(\emptyset) = 0$, and assume that for $j = 1, 2$ there exists $g_j : \mathbb{N} \cup \{0, \infty\} \to \mathbb{R}_+$, satisfying $g_j(1) > 0$, $j = 1, 2$, and $v_0(A) = g_1(\sharp((2\mathbb{N} - 1) \cap A)) + g_2(\sharp((2\mathbb{N}) \cap A))$, $A \subset \mathbb{N}$. Then v_0 is continuous if and only if*

$$\lim_{n\to\infty} g_j(n) = g_j(\infty),\ j = 1, 2, \tag{8}$$

and then (6) reaches the limit at $n = 1$, i.e., $v_2 = v_1$. ◇

With the assumptions in Corollary 3, g_1 and g_2 are non-decreasing and we have $v_0(\mathbb{N}) = g_1(\infty) + g_2(\infty)$, $v_0(2\mathbb{N} - 1) = g_1(\infty) < v_0(\mathbb{N})$ and $v_0(2\mathbb{N}) = g_2(\infty) < v_0(\mathbb{N})$, while we have $v_2 = v_1$ as in Theorem 2. We give a proof of Theorem 2 (which is easily applicable to Corollary 3) in § 2.

Both Theorem 2 and [6, Proposition 5.4] assume the initial set functions to depend only on the cardinality of the set variable. Study of general v_0 without restrictions seems less easy. In [6, Corollary 5.3], we proved that if $m = 3$ ($\sharp\Omega = 3$), then (6) reaches the limit with 1 iteration for any initial non-decreasing set function v_0, i.e., $v_2 = v_1$ so that v_1 is submodular. We show in § 3 a corresponding result for $m = 4$ such that (6) reaches the limit with at most 2 iterations, so that v_2 is submodular. (Incidentally, we gave an explicit example of $v_1 \neq v_2$ at the end of [6], hence the result in § 3 is the best possible general result for $m = 4$.) Basic idea of the proof in § 3 is classifying all the possibilities of v_0, concluding that $v_3 = v_2$ is inevitable. Though we lack proof applicable for all $m \in \mathbb{N}$ at once, we could conjecture that the recursion (6) reaches the limit after finite number of iterations on finite total sets, which partly motivates starting a study of the recursion on infinite total sets in this paper.

The present paper mainly focuses on the set functions. In [6, §3] we defined in our framework the Choquet integrations in terms of set functions, and to see that our definition provides sufficient mathematical framework for existing mathematical studies and their applications, we reproduced in [6, §4] several baseic formulas in mathematical finance corresponding to those in [7]. We continue the attempt and give a formula in § 4, for which we only had upper bound in [6].

2 Proof of Theorem 2

A proof below basically follows the proof of [6, Proposition 5.4] for the case of finite total set, but since the latter uses a natural one-to-one correspondence between $\mathcal{I} \in \mathcal{X}$ and an element in the permutation of the elements of the total set, taking advantage of the finiteness of the total set, we have to change the notation totally, and add some reasoning to take care of the infinity.

Proof of Theorem 2. Note first that the assumption $v_0(A) = g(\sharp A)$, together with $v_0(\emptyset) = 0$, $v_0(\mathbb{N}) \in \mathbb{R}_+$, and $v_0(A) \leqq v_0(B)$, $A \subset B$, implies $g(0) = 0$, $g(\infty) = g(\mathbb{N}) \in \mathbb{R}_+$, and g is non-decreasing, so that $g(n) - g(n = 1) \geqq 0$, $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} g(n) \leqq g(\infty) \in \mathbb{R}_+. \tag{9}$$

Put $\mathcal{I}_0 = \{\emptyset, \mathbb{N}\} \cup \{\{1, 2, \ldots, n\} \mid n \in \mathbb{N}\}$ Since $\sigma[\mathcal{I}] = 2^{\mathrm{N}}$ we have $\mathcal{I}_0 \in \mathcal{X}$, hence in particular, $\mathcal{X} \neq \emptyset$ and we can apply all the results in [5, 6], including Theorem 1.

Note that $\mu_{v_0,\mathcal{I}_0}$ in (3) satisfies $\mu_{v_0,\mathcal{I}_0}(I) = v_0(I)$, $I \in \mathcal{I}_0$. If v_0 is continuous, then an elementary equation $\bigcup_{n\in\mathrm{N}} \{1, \ldots, n\} = \mathbb{N}$ implies

$$\lim_{n\to\infty} g(n) = \lim_{n\to\infty} v_0(\{1, \ldots, \}) = \lim_{n\to\infty} \mu_{v_0,\mathcal{I}_0}(\{1, \ldots, \}) = \mu_{v_0,\mathcal{I}_0}(\mathbb{N}) = v_0(\mathbb{N}) = g(\infty),$$

hence (7) holds.

Conversly, assume (7). Let $\mathcal{I} \in \mathcal{X}$ and let $\mathcal{J}$ be the finitely additive class generated by $\mathcal{I}$. $\mu_{v_0,\mathcal{I}} : \mathcal{J} \to \mathbb{R}_+$ defined by (3) is a finitely additive measure on $\mathcal{J}$. To prove continuity of $\mu_{v_0,\mathcal{I}}$, let $A_n \in \mathcal{J}$, $n \in \mathbb{N}$, satisfy $A_1 \subset A_2 \subset \cdots$, and put $A = \bigcup_{n\in\mathrm{N}} A_n$, and assume $A \in \mathcal{J}$.

If $\sharp A < \infty$ (finite subset) then there exists $n_0 \in \mathbb{N}$ such that $A_n = A$, $n \geqq n_0$, hence

$$\lim_{n\to\infty} \mu_{v_0,\mathcal{I}}(A_n) = \mu_{v_0,\mathcal{I}}(A_{n_0}) = \mu_{v_0,\mathcal{I}}(A).$$

Therefore we assume $\sharp A = \infty$ in the rest of the proof for continuity of $\mu_{v_0,\mathcal{I}}$.

As in [5, (3.1)] or [6, (12)], $\mathcal{J}$ has an explicit representation

$$\begin{aligned}\mathcal{J} = \{&\bigcup_{i=1}^{k}(C_i \cap D_i^c) \mid C_1 \supset D_1 \supset C_2 \supset \cdots \supset D_k\,,\\ &C_i,\ D_i \in \mathcal{I}\,,\ i = 1,2,\ldots,k,\quad k = 1,2,3,\ldots\}.\end{aligned} \tag{10}$$

If we put $A = \bigcup_{i=1}^{k}(C_i \cap D_i^c)$, then for $n \in \mathbb{N}$, $A_n = \bigcup_{i=1}^{k}(A_n \cap C_i \cap D_i^c)$, and

$$\bigcup_{n\in\mathrm{N}}(A_n \cap C_i \cap D_i^c) = A \cap C_i \cap D_i^c = C_i \cap D_i^c,$$

hence we can consider $C_i \cap D_i^c$ for each i seperately. Therefore we assume $A = C \cap D^c$ for some $C, D \in \mathcal{I}$ satisfying $C \supset D$ in the rest of the proof for continuity of $\mu_{v_0,\mathcal{I}}$.

We are assuming $\sharp A = \infty$, hence $\sharp C = \infty$. If $\sharp D = \infty$, then monotonicity and non-negativity of the finitely additive measure $\mu = \mu_{v_0,\mathcal{I}} :\ \mathcal{J} \to \mathbb{R}_+$ imply

$$0 \leqq \mu(A_1) \leqq \mu(A_2) \leqq \cdots \leqq \mu(A) = \mu(C) - \mu(D) = v_0(C) - v_0(D) = g(\infty) - g(\infty) = 0,$$

hence $\lim_{n\to\infty} \mu_{v_0,\mathcal{I}}(A_n) = 0 = \mu_{v_0,\mathcal{I}}(A)$. Thus the proof is reduced to the case $A = C \cap D^c$ with $C, D \in \mathcal{I}$, $\sharp C = \infty$, $\sharp D < \infty$, and $C \supset D$. If there exists $n_0 \in \mathbb{N}$ and $I \in \mathcal{I}$ such that $\sharp I = \infty$ and $A_{n_0} \supset I \cap D^c$, then a similar argument as before proves

$$\begin{aligned}\lim_{n\to\infty} \mu_{v_0,\mathcal{I}}(A_n) \geqq \mu_{v_0,\mathcal{I}}(A_{n_0}) \geqq \mu_{v_0,\mathcal{I}}(I \cap D^c) &= v(I) - v(D) = g(\infty) - g(\sharp D)\\ &= v(C) - v(D) = \mu_{v_0,\mathcal{I}}(A).\end{aligned}$$

Therefore we may further assume $\sharp I < \infty$ for all A_n and $I \in \mathcal{I}$ satisfying $A_n \supset I \cap D^c$. On the other hand $C \cap D^c = A = \bigcup_{n\in\mathrm{N}} A_n$ and $\sharp C = \infty$ imply that for any $k \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ and $I \in \mathcal{I}$ such that $\sharp I \geqq k$ and $A_{n_0} \supset I \cap D^c$. Therefore

$$\mu_{v_0,\mathcal{I}}(A_{n_0}) \geqq \mu_{v_0,\mathcal{I}}(I) - \mu_{v_0,\mathcal{I}}(D) = v(I) - v(D) = g(k) - g(\sharp D),$$

which, with (7), further implies,

$$\lim_{n\to\infty} \mu_{v_0,\mathcal{I}}(A_n) \geqq \lim_{k\to\infty} g(k) - g(\sharp D) = g(\infty) - g(\sharp D) = v(C) - v(D) = \mu_{v_0,\mathcal{I}}(A).$$

This proves the continuity of $\mu_{v_0,\mathcal{I}}$ on $\mathcal{J}$.

We move on to a proof of $v_1 = v_2$ for the recursion (6) with an initial set function $v_0 :\ 2^{\mathrm{N}} \to \mathbb{R}_+$, when (7) holds.

Let $\mathcal{I} \in \mathcal{X}$. Since (7) holds by assumption, the finitely additive measure $\mu_{v_0,\mathcal{I}}$ is uniquely extended to a measure on $\sigma[\mathcal{I}] = \mathcal{F} = 2^{\mathrm{N}}$, for which we keep the same notation and write $\mu_{v_0,\mathcal{I}} : 2^{\mathrm{N}} \to \mathbb{R}_+$. Since $\mathcal{I} \subset 2^{\mathrm{N}}$ is a chain satisfying $\sigma[\mathcal{I}] = 2^{\mathrm{N}}$, for each $i \in \mathbb{N}$ there exists $I_{\mathcal{I}}(i) \in \mathcal{I}$ satisfying $i \notin I_{\mathcal{I}}(i)$ and $I_{\mathcal{I}}(i) \cup \{i\} \in \mathcal{I}$. Therefore we have

$$\mu_{v_0,\mathcal{I}}(\{i\}) = \mu_{v_0,\mathcal{I}}(I_{\mathcal{I}}(i) \cup \{i\}) - \mu_{v_0,\mathcal{I}}(I_{\mathcal{I}}(i)) = v_0(I_{\mathcal{I}}(i) \cup \{i\}) - v_0(I_{\mathcal{I}}(i)),\ i \in \mathbb{N}.$$

For $A \subset \mathbb{N}$ we then have

$$\begin{aligned}\mu_{v_0,\mathcal{I}}(A) &= \sum_{i\in A} \mu_{v_0,\mathcal{I}}(\{i\}) = \sum_{i\in A}(v_0(I_{\mathcal{I}}(i) \cup \{i\}) - v_0(I_{\mathcal{I}}(i))) \\ &= \sum_{i\in A}(g((\sharp I_{\mathcal{I}}(i)) + 1) - g(\sharp I_{\mathcal{I}}(i))),\end{aligned}$$

which further implies, with (6),

$$v_1(A) = \sup_{\mathcal{I}\in\mathcal{X}} \sum_{i\in A}(g((\sharp I_{\mathcal{I}}(i)) + 1) - g(\sharp I_{\mathcal{I}}(i))),\ A \in 2^{\mathrm{N}}. \tag{11}$$

On the other hand, since $g(n) - g(n-1) \geqq 0$ and $g(n) \leqq g(\infty) \in \mathbb{R}_+$ for all $n \in \mathbb{N}$, we can rearrange $\{g(n) - g(n-1) \mid n \in \mathbb{N}\}$ in the non-increasing order. Therefore there exists α_i, $i = 1, 2, \ldots$, such that $\{\alpha_i \mid i \in \mathbb{N}\} = \mathbb{N}$, $\alpha_i \neq \alpha_j$, $i \neq j$, and

$$g(\alpha_1) - g(\alpha_1 - 1) \geqq g(\alpha_2) - g(\alpha_2 - 1) \geqq \cdots \geqq g(\alpha_i) - g(\alpha_i - 1) \cdots. \tag{12}$$

Therefore the supremum in the right hand side of (11) is attained for $\mathcal{I} \in \mathcal{X}$ which satisfies

$$\{\alpha_j \mid j = 1, \ldots, \sharp A\} = \{(\sharp I_{\mathcal{I}}(i)) + 1 \mid i \in A\},$$

so that we have

$$v_1(A) = \sum_{j=1}^{\sharp A}(g(\alpha_j) - g(\alpha_j - 1)).$$

Note that the right hand side depends only on A and g, and does not depend on v_0. Hence if we replace v_0 by v_1 in the recursion (6) we have, with the same reasoning as above,

$$v_2(A) = \sum_{j=1}^{\sharp A}(g(\alpha_j) - g(\alpha_j - 1)),$$

which proves $v_1 = v_2$. □

3 Recursion relation for total set with 4 elements

In this section we prove the following.

Theorem 4 *Let $(\Omega, \mathcal{F}) = (\Omega_4, 2^{\Omega_4})$ be a measurable space satisfying $\sharp\Omega_4 = 4$, and $v_0 : 2^{\Omega_4} \to \mathbb{R}_+$ a non-decreasing set function satisfying $v_0(\emptyset) = 0$. Then the sequence of set functions $v_n : 2^{\Omega_4} \to \mathbb{R}_+$, $n \in \mathbb{N}$, determined by the recursion relation (6) satisfies $v_2 = v_3 = \cdots$, hence in particular, v_2 is a submodular function.* $\diamond$

Before we prove Theorem 4, we prepare some basic notations and results, partly shared with [6, §5].

For a non-decreasing continuous set function $v : \mathcal{F} \to \mathbb{R}$ satisfying $v(\emptyset) = 0$ on a measurable space $(\Omega, \mathcal{F})$ satisfying $\mathcal{X} \neq \emptyset$, we put

$$R(v) := \sup_{\mathcal{I} \in \mathcal{X}} \mu_{v,\mathcal{I}}, \tag{13}$$

which is a non-decreasing set function and satisfies $R(v)(\emptyset) = 0$.

As noted in the Introduction, if Ω is a finite set then (since all the non-decreasing set functions are continuous) $R(v)$ is also continuous, and the recursion (6) can be written as $v_{n+1} = R(v_n)$, $n = 0, 1, 2, \ldots$. Then [6, Proposition 5.1] implies that $R(v)(A) \geqq v(A)$ for each measurable set A.

We next follow the notation in the proof of [6, Proposition 5.4] valid for finite total set $\Omega = \Omega_m$. Let S_m denote the collection of the rearrangements of the sequence of m elements of the total set Ω_m;

$$S_m = \{(\sigma_1, \ldots, \sigma_m) \mid \{\sigma_1, \ldots, \sigma_m\} = \Omega_m\}, \tag{14}$$

and for $\sigma = (\sigma_1, \ldots, \sigma_m) \in S_m$ put

$$\mathcal{I}_\sigma = \{\emptyset, \{\sigma_1\}, \{\sigma_1, \sigma_2\}, \ldots, \{\sigma_1, \ldots, \sigma_{m-1}\}, \Omega_m\}. \tag{15}$$

We then have

$$\mathcal{X} = \{\mathcal{I}_\sigma \mid \sigma \in S_m\}, \tag{16}$$

and

$$\mu_{v,\mathcal{I}_\sigma}(\{\sigma_i\}) = v(\{\sigma_1, \ldots, \sigma_i\}) - v(\{\sigma_1, \ldots, \sigma_{i-1}\}), \quad i = 1, \ldots, m. \tag{17}$$

Since $S_m \ni \sigma \mapsto \mathcal{I}_\sigma \in \mathcal{X}$ determines a bijection, we identify $\mathcal{X}$ with S_m and for notational simplicity, write, for example, $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathcal{X}$ and $\mu_{v,\sigma} := \mu_{v,\mathcal{I}_\sigma}$ in the following. We also use $\vee$ and $\bigvee$, where $a \vee b = a \Leftrightarrow a \geqq b$ and $\bigvee_{i=1}^{n} a_i = \bigvee_{i \in \{1,\ldots,n\}} a_i = \max_{i \in \{1,\ldots,n\}} a_i$.

Define $\sigma^{-1} : \Omega_m \to \Omega_m$ by

$$\sigma_{\sigma^{-1}(i))} = i, \ i \in \Omega_m. \tag{18}$$

Then (13) and (17) imply

$$\mu_{v,\sigma}(A) = \sum_{i \in A} (v(\{\sigma_1, \ldots, \sigma_{\sigma^{-1}(i)-1}, i\}) - v(\{\sigma_1, \ldots, \sigma_{\sigma^{-1}(i)-1}\})), \ A \in 2^{\Omega_m}, \ \sigma \in \mathcal{X}. \tag{19}$$

We note that the last example in [6, §5] shows that $R(v)$ is not necessarily submodular, while it is easy to show that $R(v)$ is subadditive.

Proposition 5 *Let $m \in \mathbb{N}$ and $(\Omega, \mathcal{F}) = (\Omega_m, 2^{\Omega_m})$, where $\sharp\Omega_m = m$. If $A, B \in 2^{\Omega_m}$ satisfy $A \cap B = \emptyset$, then for any non-decreasing set function $v : 2^{\Omega_m} \to \mathbb{R}$ satisfying $v(\emptyset) = 0$, it holds that $R(v)(A \cup B) \leqq R(v)(A) + R(v)(B)$.* $\diamond$

Proof. The definition (13) of R implies that there exists $\sigma \in \mathcal{X}$ such that

$$R(v)(A \cup B) = \mu_{v,\mathcal{I}_\sigma}(A \cup B) = \mu_{v,\mathcal{I}_\sigma}(A) + \mu_{v,\mathcal{I}_\sigma}(B) \leqq R(v)(A) + R(v)(B),$$

which proves the statement. $\square$

We recall the following result (with slight change of notation) from [6].

Proposition 6 ([6, Proposition 5.2]) *Let $m \in \mathbb{N}$ and Ω_m be a set of cardinality $\sharp\Omega_m = m$.*

(i) For any set function $v : 2^{\Omega_m} \to \mathbb{R}_+$ if one of the following (1)–(7) holds for a pair of sets $A, B \subset \Omega_m$, then $v(A \cap B) + v(A \cup B) = v(A) + v(B)$ holds; (1) $A = \emptyset$, or (2) $B = \emptyset$, or (3) $A = \Omega_m$, or (4) $B = \Omega_m$, or (5) $A \subset B$, or (6) $A \supset B$, or (7) $A = B$.

(ii) Let $v : 2^{\Omega_m} \to \mathbb{R}$ be a non-decreasing set function satisfying $v(\emptyset) = 0$. If one of the following (1)–(4) holds for a pair of sets $A, B \subset \Omega_m$, then $R(v)(A \cap B) + R(v)(A \cup B) \leqq R(v)(A) + R(v)(B)$ holds; (1) $A \cap B = \emptyset$, (2) $A \cup B = \Omega_m$, (3) $\sharp A = 1$, (4) $\sharp A = m - 1$. $\diamond$

A proof is in the reference.

Corollary 7 *Let $m \in \mathbb{N}$. If $v : 2^{\Omega_m} \to \mathbb{R}$ is a non-decreasing set function satisfying $v(\emptyset) = 0$, the following holds.*

(i) For $A \in 2^{\Omega_m}$ satisfying $\sharp A = 1$, $R^2(v)(A) = R(v)(A)$.

(ii) If $m \geqq 2$, then for $A \in 2^{\Omega_m}$ satisfying $\sharp A = m - 1$, $R^2(v)(A) = R(v)(A)$. $\diamond$

Proof. (i) Without loss of generality we may write $\Omega_m = \{1, \ldots, m\}$ and $A = \{1\}$. Then (13) and (19) imply, by classifying the sequence $\sigma = (\sigma_1, \ldots, \sigma_m) \in S_m$ by the position of the element 1,

$$\begin{aligned} R^2(v)(\{1\}) &= \bigvee_{\sigma \in \mathcal{X}} (R(v)(\{\sigma_1, \ldots, \sigma_{\sigma^{-1}(1)-1}, 1\}) - R(v)(\{\sigma_1, \ldots, \sigma_{\sigma^{-1}(1)-1}\})) \\ &= R(v)(\{1\}) \vee \bigvee_{\emptyset \subsetneqq C \subset \{1\}^c} (R(v)(C \cup \{1\}) - R(v)(C)). \end{aligned}$$

Since Proposition 5 implies $R(v)(C \cup \{1\}) \leqq R(v)(C) + R(v)(\{1\})$, when $C \cap \{1\} = \emptyset$, we have $R^2(v)(\{1\}) = R(v)(\{1\})$.

(ii) Without loss of generality we may write $\Omega_m = \{1, \ldots, m\}$ and $A = \{1, \ldots, m-1\} = \{m\}^c$. As in the above proof, (13) imply

$$R^2(v)(\{m\}^c) = R(v)(\{m\}^c) \vee \bigvee_{\emptyset \subsetneqq C \subsetneqq \{m\}^c} (R(v)(\Omega_m) - R(v)(C \cup \{m\}) + R(v)(C)). \tag{20}$$

Put $A = \{m\}^c$ and $B = C \cup \{m\}$ in the second term in the right hand side of (20). Then $A \cup B = \Omega_m$, hence Proposition 6(ii)(2) implies

$$R(v)(C) + R(v)(\Omega_m) \leqq R(v)(\{m\}^c) + R(v)(C \cup \{m\}),$$

hence (20) implies $R^2(v)(\{m\}^c) = R(v)(\{m\}^c)$. □

Proof of Theorem 4. Theorem 1 implies that it suffices to prove $R^3(v)(A) = R^2(v)(A)$, $A \subset \Omega_4$, for all non-decreasing set function $v : 2^{\Omega_4} \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. Corollary 7 proves the cases $\sharp A = 1$ and $\sharp A = 3$. The cases $\sharp A = 0$ ($A = \emptyset$) and $\sharp A = 4$ ($A = \Omega_4$) are obvious, hence it only remains to prove the case $\sharp A = 2$. Without loss of generality we may therefore put $\Omega_4 = \{1, 2, 3, 4\}$ and $A = \{1, 2\}$ in the rest of the proof.

As in the proof of Corollary 7, (13) and (19) imply

$$\begin{aligned}
&R^3(v)(\{1,2\}) \\
&= \bigvee_{j=1}^{2} \bigvee_{D \subset C \subset \{3,4\}} (R^2(v)(\{1,2\} \cup C) - R^2(v)(\{j\} \cup C) + R^2(v)(\{j\} \cup D) - R^2(v)(D)) \\
&= R^2(v)(\{1,2\}) \vee \bigvee_{C \subset \{3,4\}} (R^2(v)(\{1,2\} \cup C) - R^2(v)(C)) \\
&\vee \bigvee_{j=1}^{2} (R^2(v)(\Omega_4) - R^2(v)(\{j,3,4\}) + R^2(v)(\{j\})) \\
&\vee \bigvee_{j=1}^{2} \bigvee_{k=3}^{4} (R^2(v)(\Omega_4) - R^2(v)(\{j,3,4\}) + R^2(v)(\{j,k\}) - R^2(v)(\{k\})) \\
&\vee \bigvee_{j=1}^{2} \bigvee_{k=3}^{4} (R^2(v)(\{1,2,k\}) - R^2(v)(\{j,k\}) + R^2(v)(\{j\})).
\end{aligned} \tag{21}$$

Proposition 5 implies $R^2(v)(\{1,2\} \cup C) - R^2(v)(C) \leqq R^2(v)(\{1,2\})$ for $C \subset \{3,4\}$. Proposition 6(ii)(2) with $m = 4$, $A = \{1,2\}$, and $B = \{j,3,4\}$ implies

$$R(v)(\Omega_4) - R(v)(\{j,3,4\}) + R(v)(\{j\}) \leqq R(v)(\{1,2\}),\ j = 1,2.$$

Similarly, Proposition 6(ii)(2) and Proposition 5 imply

$$\begin{aligned}
&R(v)(\Omega_4) - R(v)(\{j,3,4\}) + R(v)(\{j,k\}) - R^2(v)(\{k\}) \leqq R(v)(\{1,2,k\}) - R^2(v)(\{k\}) \\
&\leqq R(v)(\{1,2\}),\ j = 1,2,\ k = 3,4.
\end{aligned}$$

Note that the equation corresponding to (21) for $R^2(v)(\{1,2\})$ instead of $R^3(v)(\{1,2\})$ implies

$$R(v)(\{1,2,k\}) - R(v)(\{j,k\}) + R(v)(\{j\}) \leqq R^2(\{1,2\}),\ j=1,2,\ k=3,4. \tag{22}$$

Corollary 7, non-decreasing propoerty of R (i.e., $R^2(v)(A) \geqq R(v)(A)$), and (22) imply

$$\begin{aligned} &R^2(v)(\{1,2,k\}) - R^2(v)(\{j,k\}) + R^2(v)(\{j\}) \\ &= R(v)(\{1,2,k\}) - R^2(v)(\{j,k\}) + R(v)(\{j\}) \\ &\leqq R(v)(\{1,2,k\}) - R(v)(\{j,k\}) + R(v)(\{j\}) \leqq R^2(v)(\{1,2\}),\ j=1,2,\ k=3,4. \end{aligned}$$

Therefore (21) implies $R^3(v)(\{1,2\}) = R^2(v)(\{1,2\})$. □

4 Choquet integration version of the fundamental representation formula

In this section, we fix a measurable space $(\Omega, \mathcal{F})$ which satisfies $\mathcal{X} \neq \emptyset$.

Let $v:\ \mathcal{F} \to \mathbb{R}$ be a non-decreasing set function, satisfying $v(\emptyset) = 0$, and $f:\ \Omega \to \mathbb{R}$ a real valued measurable function. Following [6, §3], we define in our framework the Choquet integration of f with respect to v by

$$v(f) = \lim_{y \to -\infty} \Big(y\, v(\Omega) + \int_y^\infty v(\{\omega \in \Omega \mid f(\omega) > z\})\, dz \Big) \tag{23}$$

whenever the right-hand side is a real value. If $v(f)$ of (23) is defined it is equal to the asymmetric integral in terms of [4, Chap. 5]. If $f(\omega) \geqq x$, $\omega \in \Omega$, holds for an $x \in \mathbb{R}$, then (23) has a simpler expression

$$v(f) = x\, v(\Omega) + \int_x^\infty v(\{\omega \in \Omega \mid f(\omega) > z\})\, dz. \tag{24}$$

Simple function, a measurable function whose image is a finite set, is an elementary example of Choquet integrable function. Following [6, §3], we adopt a representation $f = \sum_{i=1}^n a_i\, \mathbf{1}_{A_i}$, where

$$\begin{aligned} &a_i \geqq 0,\ i = 1, \ldots, n-1,\ a_n \in \mathbb{R}, \\ &\emptyset = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset A_n = \Omega. \end{aligned}$$

Here and in the following, we use a symbol $\mathbf{1}_A$ to denote a characteristic function of a set A defined by $\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$ (We use $\mathbf{1}_\cdot$ on any space.) We then have $v(f) = \sum_{i=1}^n a_i\, v(A_i)$.

In [6], we also formulated an analog of monotone convergence theorem for the Choquet integration. To state the theorem, we extend some elementary terms in measure theory to

continuous, non-decreasing set function v. We say that a measurable set $N \in \mathcal{F}$ is a v-null set, if

$$v(N \cup A) = v(N^c \cap A) = v(A),\ A \in \mathcal{F}. \tag{25}$$

For a sequence of measurable functions $f_n :\ \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, and a measurable function $f :\ \Omega \to \mathbb{R}$, we say that the sequence converges v-almost surely to f, if there exists a v-null set $N \in \mathcal{F}$ such that

$$\lim_{n\to\infty} f_n(\omega) = f(\omega),\ \omega \in N^c. \tag{26}$$

Theorem 8 ([6, Theorem 3.1]) *Let $(\Omega, \mathcal{F})$ be a measurable space satisfying $\mathcal{X} \neq \emptyset$, and $v :\ \mathcal{F} \to \mathbb{R}$ be a non-decreasing, continuous set function, satisfying $v(\emptyset) = 0$. If $f_n :\ \Omega \to \mathbb{R}_+$, $n \in \mathbb{N}$, and $f :\ \Omega \to \mathbb{R}$ are measurable and Choquet integrable non-negative valued functions such that f_n are pointwise non-decreasing in n and $f = \lim_{n\to\infty} f_n$, v-almost surely, then $\lim_{n\to\infty} v(f_n) = v(f)$ holds.* ◇

As in standard measure theory, a simple function approximation of a non-negative Choquet integrable function $f :\ \Omega \to \mathbb{R}_+$ defined by

$$f_k := \sum_{i=1}^{k\,2^k} 2^{-k}(i-1)\,\mathbf{1}_{2^{-k}(i-1)<f5 2^{-k}i} \ +k\,\mathbf{1}_{\{\omega\in\Omega | f(\omega)>k\}},\quad k \in \mathbb{N}, \tag{27}$$

is an example of non-decreasing sequence of functions which converges pointwise to f as $k \to \infty$. Theorem 8 then implies $v(f) = \lim_{k\to\infty} v(f_k)$.

We refer to [6, §3] for proofs and other details on basic properties.

If we put $\rho(f) = \dfrac{v(-f)}{v(\Omega)}$ then ρ is a coherent risk measure in mathematical finance. Our framework does not introduce probability measures, while assumes $\mathcal{X} \neq \emptyset$. In contrast, [7] assumes a standard probability space with non-atomic probability measure. See [5, §1,§4] for more on discussion and open problems on measurable spaces with $\mathcal{X} \neq \emptyset$. To see that, in spite of the difference in the basic framework, our definition provides a mathematical framework for practical application to mathematical finance, we reproduced in [6, §4] several baseic formulas corresponding to those in [7]. The reference considers law invariant coherent risk measures, which we formulated in our framework as follows.

Let $(\Omega, \mathcal{F})$ be a measurable space satisfying $\mathcal{X} \neq \emptyset$, and denote the set of finite measures on $(\Omega, \mathcal{F})$ by $\mathcal{M}(\Omega)$.

Theorem 9 ([6, Theorem 4.2]) *For $\mu \in \mathcal{M}(\Omega)$ satisfying $\mu \ll \nu$, denote the Radon–Nykodim derivative by $\dfrac{d\,\mu}{d\nu} :\ \Omega \to \mathbb{R}_+$, and define the distribution function $F_\mu :\ \mathbb{R}_+ \to \mathbb{R}_+$ by*

$$F_\mu(y) = \nu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) \leqq y\}),\ y \in \mathbb{R}_+, \tag{28}$$

and define a set function $v_\mu :\ \mathcal{F} \to \mathbb{R}$ by

$$v_\mu(A) = \int_0^\infty (\nu(\Omega) - F_\mu(z)) \wedge \nu(A)\,dz,\ A \in \mathcal{F}. \tag{29}$$

Then the following hold.

(i) v_μ *is non-decreasing, continuous, submodular, and satisfies* $v_\mu(\emptyset) = 0$.

(ii) It holds that $v_\mu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y\}) = \mu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > y\}),\ y \in \mathbb{R}_+$.

(iii) For a non-negative valued measurable function $f : \Omega \to \mathbb{R}_+$, *let* $v_\mu(f)$ *denote the Choquet integration of* f *with respect to* v_μ *in (24). Then*

$$v_\mu(f) \geqq \sup\{\int_\Omega f(\omega)\mu'(d\omega) \mid \mu' \in \mathcal{M}(\Omega),\ \mu' \ll \nu,\ F_{\mu'} = F_\mu\} \tag{30}$$

holds. In particular, it holds that

$$v_\mu(A) \geqq \sup\{\mu'(A) \mid \mu' \in \mathcal{M}(\Omega),\ \mu' \ll \nu,\ F_{\mu'} = F_\mu\},\ A \in \mathcal{F}. \tag{31}$$

◇

See [6, §4] for a proof.

In the right hand sides of (5) in Theorem 1 and (31), suprema are taken over measures, but the conditions are different. Therefore, equality in place of inequality in (30) may need extra assumptions. In fact, a proof of the corresponding equality result in [7] uses the assumption of standard probability space with non-atomic probability measure, in order to reduce the proof to the case of the 1-dimensional Lebesgue measure on a unit interval $[0, 1)$. We have a following result for equality version of (31).

We recall a following definition in [5, §2] and [6, §1 and §2] of insertion of a countable chain into $\mathcal{I} \in \mathcal{X}$. For $\mathcal{I} \in \mathcal{X}$ and $A \in \mathcal{F}$, define $\mathcal{I}_A$, the insertion of A into $\mathcal{I}$, by

$$\mathcal{I}_A = \{A \cap I \mid I \in \mathcal{I}\} \cup \{A \cup I \mid I \in \mathcal{I}\}. \tag{32}$$

Note that $\mathcal{X}$ is closed under the insertion; if $\mathcal{I} \in \mathcal{X}$ and $A \in \mathcal{F}$ then $A \in \mathcal{I}_A \in \mathcal{X}$ [5, Lemma 2]. If $\{A, B\}$ is a chain, i.e., $A \subset B$ or $A \supset B$, then the order of insertion is irrelevant, and we will use the notation $\mathcal{I}_{A,B} := (\mathcal{I}_A)_B = (\mathcal{I}_B)_A$. For example, if $B \subset A$ then

$$\mathcal{I}_{A,B} := (\mathcal{I}_A)_B = \{I \cap B,\ (I \cup B) \cap A,\ I \cup A \mid I \in \mathcal{I}\} = (\mathcal{I}_B)_A \in \mathcal{X}. \tag{33}$$

This insertion is inductively defined for chain of any size. (Insertion of a countably infinite chain is also defined in [6, §2].)

Theorem 10 *Let* $(\Omega, \mathcal{F})$ *be a measurable space satisfying* $\mathcal{X} \neq \emptyset$ *and let* $\mu, \nu \in \mathcal{M}(\Omega)$ *be a pair of finite measures on the space satisfying* $\mu \ll \nu$. *Assume that there exists* $\mathcal{I}^\mu \in \mathcal{X}$ *such that*

$$\{\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > z\} \mid z \geqq 0\} \subset \mathcal{I}^\mu$$

and

$$\begin{array}{l}(\forall n \in \mathbb{N})\ (\forall \mathcal{A} = \{A_i \mid i = 1, \ldots, n\} \subset \mathcal{F};\ A_1 \subset \cdots \subset A_n)\ \exists \mathcal{I}^* \in \mathcal{X};\\ \mathcal{I}^\mu_{\mathcal{A}} \subset \mathcal{I}^*,\ \mathrm{Im}(\nu|_{\mathcal{I}^*}) = [0, \nu(\Omega)],\end{array} \tag{34}$$

where $\mathcal{I}^\mu_{\mathcal{A}}$ *is the insertion of the chain* $\mathcal{A}$ *into* $\mathcal{I}^\mu$.

Then for each $n \in \mathbb{N}$ and $\mathcal{A} = \{A_i \mid i = 1, \dots, n\} \subset \mathcal{F}$ satisfying $A_1 \subset \cdots \subset A_n$ there exists $\mu' \in \mathcal{M}(\Omega)$ such that

$$v_\mu(A_i) = \mu'(A_i) = \sup\{\mu''(A_i) \mid \mu'' \in \mathcal{M}(\Omega),\ \mu'' \ll \nu,\ F_{\mu''} = F_\mu\},\ A_i \in \mathcal{A}, \tag{35}$$

where v_μ is as in (29), and the distribution functions F_μ and $F_{\mu''}$ of the measures μ and μ'' are as in (28). Furtheremore,

$$v_\mu(f) = \sup\{\int_\Omega f(\omega)\mu'(d\omega) \mid \mu' \in \mathcal{M}(\Omega),\ \mu' \ll \nu,\ F_{\mu'} = F_\mu\} \tag{36}$$

holds for each bounded measurable function $f:\ \Omega \to \mathbb{R}$. ◇

Remark. In the proof below we find $\mu' \in \mathcal{M}(\Omega)$ in (35) such that $\{\{\omega \in \Omega \mid \frac{d\,\mu'}{d\nu}(\omega) > z\} \mid z \geqq 0\} \cup \mathcal{A}$ is a chain, as well as $\mu' \ll \nu$ and $F_{\mu'} = F_\mu$. ◇

Proof. To prove (35), put

$$I_{\nu(A_i)} = A_i\,,\ i = 1, \dots, n. \tag{37}$$

Assumption (34) then implies that there exists $I_\alpha \in \mathcal{I}^*$, $\alpha \in [0, \nu(\Omega)]$, such that

$$\nu(I_\alpha) = \alpha,\ \alpha \in [0, \nu(\Omega)] \cap \{\nu(A_i) \mid i = 1, \dots, n\}^c. \tag{38}$$

Define $\alpha:\ \mathbb{R}_+ \to [0, \nu(\Omega)]$ by

$$\alpha(z) = \nu(\{\omega \in \Omega \mid \frac{d\,\mu}{d\nu}(\omega) > z\}) = \nu(\Omega) - F_\mu(z),\ z \geqq 0, \tag{39}$$

and $\frac{d\,\mu'}{d\nu}:\ \Omega \to \mathbb{R}_+$ by

$$\frac{d\,\mu'}{d\nu}(\omega) = \sup\{z \geqq 0 \mid \omega \in I_{\alpha(z)}\}. \tag{40}$$

Since $\alpha(z)$ in (39) is non-increasing, $\frac{d\,\mu'}{d\nu}$ in (40) is non-negative measurable. Hence $\mu'(A) = \int_A \frac{d\,\mu'}{d\nu}(\omega)\nu(d\omega)$ defines a measure $\mu':\ \mathcal{F} \to \mathbb{R}_+$ satisfying $\mu' \ll \nu$.

We first prove $F_{\mu'} = F_\mu$ and that $\{\omega \in \Omega \mid \frac{d\,\mu'}{d\nu}(\omega) > z\} \cup \mathcal{A}$ is a chain. Since a distribution function is right continuous, (39) implies

$$\lim_{n\to\infty} \alpha(z + \frac{1}{n}) = \alpha(z),\ z \geqq 0. \tag{41}$$

Also, (39) implies that $\alpha(z)$ is non-increasing. This and (38) and $I_\alpha \in \mathcal{I}^*$ together imply that $\mathcal{I}_{\alpha(z)}$ is non-increasing in z. Hence, with (40) we have

$$I_{\alpha(z+\frac{1}{n})} \subset \{\omega \in \Omega \mid \frac{d\,\mu'}{d\nu}(\omega) > z\} \subset I_{\alpha(z)}\,,\ n \in \mathbb{N}, \tag{42}$$

which futher implies $\alpha(z+\frac{1}{n}) \leqq \nu(\Omega) - F_{\mu'}(z) \leqq \alpha(z)$ for all $n \in \mathbb{N}$. This with (39) and (41) leads to $F_{\mu'}(z) = \nu(\Omega) - \alpha(z) = F_\mu(z),\ z \geqq 0$, hence $F_{\mu'} = F_\mu$ holds. Similarly, for $i \in \{1,\dots,n\}$, if $\nu(A_i) \geqq \alpha(z)$ then we have, with (42), $\{\omega \in \Omega \mid \frac{d\,\mu'}{d\nu}(\omega) > z\} \subset I_{\alpha(z)} \subset I_{\nu(A_i)} = A_i$. On the other hand, if $\nu(A_i) < \alpha(z)$ then (41) implies that there exists an integers n such that $\nu(A_i) < \alpha(z+\frac{1}{n})$, for which $A_i \subset I_{\alpha(z+\frac{1}{n})} \subset \{\omega \in \Omega \mid \frac{d\,\mu'}{d\nu}(\omega) > z\}$. This proves that $\{\omega \in \Omega \mid \frac{d\,\mu'}{d\nu}(\omega) > z\} \cup \mathcal{A}$ is a chain. This holds for all $z \geqq 0$, which proves that $\{\{\omega \in \Omega \mid \frac{d\,\mu'}{d\nu}(\omega) > z\} \mid z \geqq 0\} \cup \mathcal{A}$ is a chan.

To prove the equality in (35), note first an elementary fact:

(43) If $\{A, B\} \subset \mathcal{F}$ is a chain, then $\nu(A) \wedge \nu(B) = \nu(A \cap B)$.

Using $F_\mu = F_{\mu'}$, the chain property of $\{\{\omega \in \Omega \mid \frac{d\,\mu'}{d\nu}(\omega) > z\} \mid z \geqq 0\} \cup \mathcal{A}$, and (43) in (29), we have

$$\begin{aligned} v_\mu(A_i) &= \int_0^\infty (\nu(\Omega) - F_\mu(z)) \wedge \nu(A_i)\, dz = \int_0^\infty (\nu(\Omega) - F_{\mu'}(z)) \wedge \nu(A_i)\, dz \\ &= \int_0^\infty \nu(A_i) \wedge \nu(\{\omega \in \Omega \mid \frac{d\,\mu'}{d\nu}(\omega) > z\})\, dz \\ &= \int_0^\infty \nu(A_i \cap \{\omega \in \Omega \mid \frac{d\,\mu'}{d\nu}(\omega) > z\})\, dz = \int_\Omega \int_0^\infty \mathbf{1}_{\frac{d\,\mu'}{d\nu}(\omega) > z}\, \mathbf{1}_{\omega \in A_i}\, dz\, \nu(d\omega) \\ &= \int_{A_i} \frac{d\,\mu'}{d\nu}(\omega)\, \nu(d\omega) = \int_{A_i} \mu'(d\omega) = \mu'(A_i) \end{aligned}$$

This proves (35).

We move on to a proof of (36). By assumption f in (36) is bounded, hence we may add a constant if necessary so that we may assume from the beginning that f is non-negative valued. Let f_k, $k \in \mathbb{N}$, be the simple function approximation (27) for f. Then (27) and Theorem 8 imply that for any positive ϵ, there exists an integer k_0 such that for any integer $k > k_0$ it holds that $f_k(\omega) + \epsilon \geqq f(\omega) \geqq f_k(\omega)$. $\omega \in \Omega$, and $v(f_k) + \epsilon \geqq v(f) \geqq v(f_k)$. Take k_0 sufficiently large so that $|f(\omega)| \leqq k_0$ also holds.

Let $k > k_0$ and put $n = k\,2^k,\ a_i = 2^{-k},\ i = 1,\dots,n-1$, and

$$\mathcal{A} = \{A_i \mid i = 1,\dots,n\}; A_i = \{\omega \in \Omega \mid f(\omega) \geqq \sum_{j=i}^n a_j = a_n + (n-i)2^{-k}\} \in \mathcal{F},\ i = 1,\dots,n.$$

Then we have $A_1 \subset A_2 \subset \cdots \subset A_n = \Omega$ and $f_k = \sum_{i=1}^n a_i\, \mathbf{1}_{A_i}$. Using also (35), we have

$$v_\mu(f_k) = \sum_{i=1}^n a_i\, v(A_i) = 2^{-k} \sum_{i=1}^{n-1} v_\mu(A_i) + a_n \mu(\Omega) = 2^{-k} \sum_{i=1}^{n-1} \mu'(A_i) + a_n \mu'(\Omega) = \int_\Omega f_k(\omega) \mu'(d\omega).$$

Therefore,

$$\begin{aligned} v_\mu(f) &\leqq v_\mu(f_k) + \epsilon = \int_\Omega f_k(\omega)\mu'(d\omega) + \epsilon \\ &= \sup\{\int_\Omega f_k(\omega)\mu'(d\omega) \mid \mu' \in \mathcal{M}(\Omega),\ \mu' \ll \nu,\ F_{\mu'} = F_\mu\} + \epsilon \\ &\leqq \sup\{\int_\Omega f(\omega)\mu'(d\omega) \mid \mu' \in \mathcal{M}(\Omega),\ \mu' \ll \nu,\ F_{\mu'} = F_\mu\} + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this and (30) iimply (36). □

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