Triviality of hierarchical Ising model in four dimensions

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Abstract

Existence of critical renormalization group trajectory for a hierarchical Ising model in 4 dimensions is shown. After 70 iterations of renormalization group transformations, the critical Ising model is mapped into a vicinity of the Gaussian fixed point. Convergence of the subsequent trajectory to the Gaussian fixed point is shown by power decay of the effective coupling constant. The analysis in the strong coupling regime is computer-aided and Newman's inequalities on truncated correlations are used to give mathematical rigor to the numerical bounds. In order to obtain a criterion for convergence to the Gaussian fixed point, characteristic functions and Newman's inequalities are systematically used.

1 Introduction and main result.

Dyson's Hierarchical spin system is an equilibrium statistical mechanical system defined as follows [4, 16, 3, 6, 14]. Let Λ be a positive integer, and denote the 2^Λ variables (spin variables) ϕ_θ, Hamiltonian H_Λ, and the expectation values $\langle\cdot\rangle$, respectively, by

$$\begin{aligned}
\phi_\theta &= \phi_{\theta_\Lambda,\dots,\theta_1}\,, \quad \theta=(\theta_\Lambda,\dots,\theta_1)\in\{0,1\}^\Lambda,\\
H_\Lambda(\phi) &= -\frac{1}{2}\sum_{n=1}^{\Lambda}\left(\frac{c}{4}\right)^n \sum_{\theta_\Lambda,\dots,\theta_{n+1}}\left(\sum_{\theta_n,\dots,\theta_1}\phi_{\theta_\Lambda,\dots,\theta_1}\right)^2,\\
\langle F\rangle_{\Lambda,h} &= \frac{1}{Z_{\Lambda,h}}\int d\phi F(\phi)\exp(-\beta H_\Lambda(\phi))\prod_\theta h(\phi_\theta),\\
Z_{\Lambda,h} &= \int d\phi\exp(-\beta H_\Lambda(\phi))\prod_\theta h(\phi_\theta),
\end{aligned}$$

where h is a single spin measure density normalized as

$$\int_{\mathbb{R}} h(x)dx = 1.$$

In the following, we shall fix the so far arbitrary normalization of the spin variables by

$$\beta = \frac{1}{c} - \frac{1}{2}\,. \tag{1.1}$$

Hierarchical models are so designed that the block-spin renormalization group transformation $\mathcal{R}$ has a simple form. In fact, $\mathcal{R}$ is a non-linear transformation of functions on $\mathbb{R}$, defined as follows. Define the block spins ϕ' by

$$\phi'_\tau = \frac{\sqrt{c}}{2} \sum_{\theta_1=0,1} \phi_{\tau\theta_1} \, , \ \tau = (\tau_{\Lambda-1}, ..., \tau_1).$$

If a function $F(\phi)$ depends on ϕ through ϕ' only, namely, if there is a function $F'(\phi')$ on the block spins such that

$$F(\phi) = F'(\phi'),$$

then it holds that

$$\langle F \rangle_{\Lambda,h} = \langle F' \rangle_{\Lambda-1,\mathcal{R}h} \, ,$$

where

$$\mathcal{R}h(x) = \text{const.} \exp(\frac{\beta}{2}x^2) \int_{\mathbb{R}} h(\frac{x}{\sqrt{c}} + y) h(\frac{x}{\sqrt{c}} - y) \, dy, \ x \in \mathbb{R}. \tag{1.2}$$

Note that

$$h_G(x) = \text{const.} \exp(-\frac{1}{4}x^2) \tag{1.3}$$

is a fixed point of $\mathcal{R}$, which we shall refer to as the density function of the massless Gaussian measure. By looking into the asymptotics of e.g., susceptibility for the hierarchical massless Gaussian model defined by (1.3), and comparing it with that of standard nearest neighbor massless Gaussian models on d-dimensional regular lattice, we see that the dimensionality d of the system may be identified (at least for the Gaussian fixed point) as

$$c = 2^{1-2/d} \qquad (\beta = \frac{1}{2}(2^{2/d} - 1)). \tag{1.4}$$

We shall extend the correspondences to hierarchical models with arbitrary measures, and use the terminology *d-dimensional hierarchical models* whenever (1.4) holds.

Asymptotic properties of the renormalization group trajectories

$$h_N = \mathcal{R}^N h_0 \, , \ N = 0, 1, 2, \cdots , \tag{1.5}$$

are extensively investigated in a 'weak coupling regime' i.e., in a 'neighborhood' of h_G [16, 3, 6, 7, 8]. In particular, it is known that, if $d \geq 4$, then there are no non-Gaussian fixed points in a 'neighborhood' of h_G, and that a 'continuum limit' constructed from a critical trajectory with an initial function in a 'neighborhood' of h_G is trivial (Gaussian).

However, in order to study asymptotic properties of strongly coupled models, we have to analyze trajectories (1.5) with initial functions in a 'strong coupling regime' far away from the Gaussian fixed point.

As a typical example, we consider in this paper the hierarchical Ising model, which is defined by the Ising spin measure density parameterized by $s \geq 0$:

$$h_{\mathrm{I},s}(x) = \frac{1}{2}(\delta(x - s) + \delta(x + s)), \tag{1.6}$$

which may be regarded as a strong coupling limit of the ϕ^4 measures:

$$h_{\mu,\lambda}(x) = \text{const.} \exp(-\mu x^2 - \lambda x^4), \quad \mu = -2\lambda s^2, \ \lambda \to \infty.$$

Here and in the following, we use the standard notation $\delta(x - s) \, dx$ denoting a probability measure with unit mass on a single point $x = s$. Hierarchical Ising model has an infinite volume limit $\Lambda \to \infty$, if $0 < c < 2$ ($d > 0$), and has a phase transition, if $1 < c < 2$ ($d > 2$) [4].

It has been widely believed without proof that the hierarchical Ising model in $d \geq 4$ dimensions has a critical trajectory converging to the Gaussian fixed point and that the 'continuum limit' of the hierarchical

Ising model in $d \geq 4$ dimensions will be trivial. In this paper, we prove this fact. In the present analysis, it is crucial that the critical Ising model is mapped into a weak coupling regime after a *small* number of renormalization group transformations (in fact, 70 iterations for $d = 4$). Moreover, using a framework essentially different from that of [16, 7], we see in the weak coupling regime that the 'effective coupling constant' of a critical model decays as $c_1/(N + c_2)$ after N iterations in $d = 4$ dimensions (exponentially for $d > 4$). Our framework in the weak coupling regime is designed especially for a critical trajectory starting at the strong coupling regime so that the criterion of convergence to the Gaussian fixed point can be checked numerically with mathematical rigor.

Corresponding results, triviality of ϕ_4^4 spin model on regular lattice ('full model'), are much far harder, and a proof of triviality of Ising model on 4 dimensional regular lattice is, though widly believed, still open. We should here note the excellent and hard works of [9, 10] where the existence of critical trajectory in the weak coupling regime (near Gaussian fixed point; 'weak triviality') is solved by rigorous block spin renormalization group transformation.

Our main theorem is the following:

Theorem 1.1 *If $d \geq 4$ (i.e. $c \geq \sqrt{2}$), there exists a 'critical trajectory' converging to the Gaussian fixed point starting from the hierarchical Ising models. Namely, there exists a positive real number s_c such that if h_N, $N = 0, 1, 2, \cdots$, are defined by (1.5) with $h_0 = h_{\mathrm{I},s_c}$, then the sequence of measures $h_N(x)\,dx$, $N = 0, 1, 2, \cdots$, converges weakly to the massless Gaussian measure $h_G(x)\,dx$.*

Remark. Our proof is partially computer-aided and shows for $d = 4$ that

$$s_c \in [1.7925671170092624, 1.7925671170092625].$$

In the following sections, we give a proof of Theorem 1.1. We will concentrate on the case $d = 4$, since the cases $d > 4$ can be proved along similar lines (with weaker bounds).

2 Strategy.

The proof of Theorem 1.1 is decomposed into two parts: Theorem 2.1(analysis in the weak coupling regime) and Theorem 2.2 (analysis in the strong coupling regime). They are stated in Section 2.3, and their proofs are given in Section 4 and Section 5, respectively. Theorem 1.1 is proved at the end of this section assuming them.

(1) In Theorem 2.1, we control the renormalization group flow in a weak coupling regime by means of a *finite* number of truncated correlations (Taylor coefficients of logarithm of characteristic functions), and, in terms of the truncated correlations, we give a criterion, a set of sufficient conditions, for the measure to be in a domain of attraction of the Gaussian fixed point.

(2) In Theorem 2.2, we prove, by rigorous computer-aided calculations, that there is a trajectory whose initial point is an Ising measure and for which the criterion in Theorem 2.1 is satisfied after a small number of iterations.

The first part (Theorem 2.1) is essentially the Bleher-Sinai argument [1, 2, 16]. However, the criteria introduced in the references [16, 7] seem to be difficult to handle when 'strong coupling constants' are present in the model, as in the Ising models. In order to overcome this difficulty, we use characteristic functions of single spin distributions and Newman's inequalities for truncated correlations.

The second part (Theorem 2.2) is basically simple numerical calculations of truncated correlations up to 8 points to ensure the criterion. The results are double checked by Mathematica and `C++` programs, and furthermore they are made mathematically rigorous by means of Newman's inequalities.

It should be noted that rigorous computer-aided proofs are employed in [14] to Dyson's hierarchical model in $d = 3$ dimensions, to prove, with [13], an existence of a non-Gaussian fixed point. (The 'physics' are of course different between $d = 3$ and $d = 4$.) We also focus on a complete mathematical proof, by combining rigorous computer-aided bounds with mathematical methods such as Newman's inequalities and the Bleher–Sinai arguments.

2.1 Characteristic function.

Denote the characteristic function of the single spin distribution h_N as

$$\hat{h}_N(\xi) = \mathcal{F}h_N(\xi) = \int_{\mathbb{R}} e^{\sqrt{-1}\xi x} h_N(x)\, dx\,. \tag{2.1}$$

The renormalization group transformation for $\hat{h}_N$ is

$$\hat{h}_{N+1} = \mathcal{F}\mathcal{R}\mathcal{F}^{-1}\hat{h}_N\,, \tag{2.2}$$

which has a decomposition

$$\mathcal{F}\mathcal{R}\mathcal{F}^{-1} = \mathcal{T}\mathcal{S}, \tag{2.3}$$

where

$$\mathcal{S}g(\xi) \quad = \quad g(\frac{\sqrt{c}}{2}\xi)^2, \tag{2.4}$$

$$\mathcal{T}g(\xi) \quad = \quad \text{const.}\ \exp(-\frac{\beta}{2}\triangle)g(\xi), \tag{2.5}$$

and the constant is so defined that

$$\mathcal{T}g\,(0) = 1\,.$$

The transformation (2.2) has same form as the $N = 2$ case of the Gallavotti hierarchical model [5, 11, 12]. Note that only for $N = 2$ the Gallavotti model is equivalent (by Fourier transform) to the Dyson's hierarchical model.

We introduce a 'potential' V_N for the characteristic function $\hat{h}_N$ and its Taylor coefficients $\mu_{n,N}$ by

$$\hat{h}_N(\xi) \quad = \quad e^{-V_N(\xi)}, \tag{2.6}$$

$$V_N(\xi) \quad = \quad \sum_{n=1}^{\infty} \mu_{n,N}\xi^n. \tag{2.7}$$

(Note that $\hat{h}_N(0) = 1$.) The coefficient $\mu_{n,N}$ is called a truncated n point correlation. They are functions of Ising parameter s in $h_0 = h_{I,s}$, but to simplify expressions, we will always suppress the dependences on s in the following.

In particular, for the initial condition $h_0 = h_{I,s}$, we have

$$\begin{aligned}
&\hat{h}_0(\xi) = \hat{h}_{I,s}(\xi) = \mathcal{F}h_{I,s}(\xi) = \cos(s\xi),\\
&\mu_{2,0} = \frac{1}{2}s^2,\ \mu_{4,0} = \frac{1}{12}s^4,\ \mu_{6,0} = \frac{1}{45}s^6,\ \mu_{8,0} = \frac{17}{2520}s^8,\ \text{etc.},
\end{aligned}$$

and

$$\begin{aligned}
&h_1(x) = \mathcal{R}h_{I,s}(x) = \text{const.}\left(e^{\beta cs^2/2}\big\{\delta(x - s\sqrt{c}) + \delta(x + s\sqrt{c})\big\} + 2\delta(x)\right),\\
&\hat{h}_1(\xi) = \frac{1}{1+k}(1 + k\cos(\sqrt{c}s\xi)), \quad \text{with} \quad k = e^{\beta cs^2/2},\\
&\mu_{2,1} = k\ell,\ \mu_{4,1} = \frac{k}{6}(2k-1)\ell^2,\ \mu_{6,1} = \frac{k}{90}(16k^2 - 13k + 1)\ell^3,\\
&\mu_{8,1} = \frac{k}{2520}(272k^3 - 297k^2 + 60k - 1)\ell^4,\ \text{etc.}, \quad \text{with} \quad \ell = \frac{cs^2}{2(k+1)}\,.
\end{aligned}$$

2.2 Newman's inequalities.

The function V_N has a remarkable positivity property and its Taylor coefficients obey Newman's inequalities (for a brief review of relevant part, see Appendix A):

$$0 \le \mu_{2n,N} \le \frac{1}{n}(2\mu_{4,N})^{n/2},\ n = 3, 4, 5, \cdots\,. \tag{2.8}$$

These inequalities follow from [15, Theorem 3, 6], since we have chosen the Ising spin distribution $h_0 = h_{I,s}$ and the function of η defined by

$$\int e^{\eta x} h_N(x)dx = \left\langle \exp\left(\eta\left(\frac{\sqrt{c}}{2}\right)^N \sum_\theta \phi_\theta\right)\right\rangle_{N,h_{I,s}} \tag{2.9}$$

has only pure imaginary zeros as is shown in [15, Theorem 1]. Note also that (1.2) and (1.6) imply

$$\mu_{2n+1,N} = 0,\ \ n = 0, 1, 2, \cdots . \tag{2.10}$$

The bounds (2.8) are extensively used in this paper. We here note the following facts:

(1) The right hand side of (2.7) has a nonzero radius of convergence.

(2) It suffices to prove $\lim_{N\to\infty} \mu_{4,N} = 0$ in order to ensure that $\mu_{2n,N}$, $n \geq 3$, converges to zero, hence the trajectory converges to the Gaussian fixed point.

2.3 Proof of Theorem 1.1.

Let $h_0 = h_{I,s}$ and $d = 4$. Note the following simple observations on the 'mass term' $\mu_{2,N}$, which is the variance of $h_N(x)\,dx$.

(1) $\mu_{2,N}$ is continuous in the Ising parameter s, because $h_N(x)\,dx$ is a result of a finite number of renormalization group transformation (1.2).

(2) $\mu_{2,N}$ is increasing in s, vanishes at $s = 0$, and diverges as $s \to \infty$.

We then put, for $N = 0, 1, 2, \cdots$,

$$\begin{aligned} \underline{s}_N &= \inf\{s > 0 \mid \mu_{2,N} \geq 1\}, & (2.11)\\ \overline{s}_N &= \inf\{s > 0 \mid \mu_{2,N} \geq \min\{1 + \frac{3}{\sqrt{2}}\mu_{4,N},\ 2+\sqrt{2}\}\}. & (2.12)\end{aligned}$$

Obviously, we have

$$0 < \underline{s}_N \leq \overline{s}_N < \infty.$$

Note also that

$$1 \leq \mu_{2,N} \leq 1 + \frac{3}{\sqrt{2}}\mu_{4,N} \tag{2.13}$$

holds for $s \in [\underline{s}_N, \overline{s}_N]$. As is seen in Section 4, (2.13) is necessary for the model to be critical. We call this *a critical mass condition.*

The following theorem states our result in the weak coupling regime and is proved in Section 4.

Theorem 2.1 *Let $h_0 = h_{I,s}$ and $d = 4$. Assume that there exist integers N_0 and N_1, satisfying $N_0 \leq N_1$, such that, for $s \in [\underline{s}_{N_1}, \overline{s}_{N_1}]$, the bounds*

$$\begin{aligned} 0 \ \leq\ & \mu_{4,N_0} \ \leq\ 0.0045, & (2.14)\\ 1.6\mu_{4,N_0}^2 \ \leq\ & \mu_{6,N_0} \ \leq\ 6.07\mu_{4,N_0}^2, & (2.15)\\ 0 \ \leq\ & \mu_{8,N_0} \ \leq\ 48.469\mu_{4,N_0}^3, & (2.16)\end{aligned}$$

and

$$\mu_{2,N} < 2 + \sqrt{2}, \quad N_0 \leq N < N_1\,, \tag{2.17}$$

hold. Then there exists an $s_c \in [\underline{s}_{N_1}, \overline{s}_{N_1}]$ such that if $s = s_c$ then

$$\lim_{N\to\infty} \mu_{4,N} = 0,$$
$$\lim_{N\to\infty} \mu_{2,N} = 1\,.$$

Remark. The original Bleher–Sinai argument takes $N_0 = N_1$. We include the $N_0 < N_1$ case which makes it possible to complete our proof by evaluating various quantities only at 2 endpoints of the interval in consideration for Ising parameter s, instead of all values in the interval, as is implicit in the assumptions of Theorem 2.1. This point will be clarified at the end of Section 5.3.

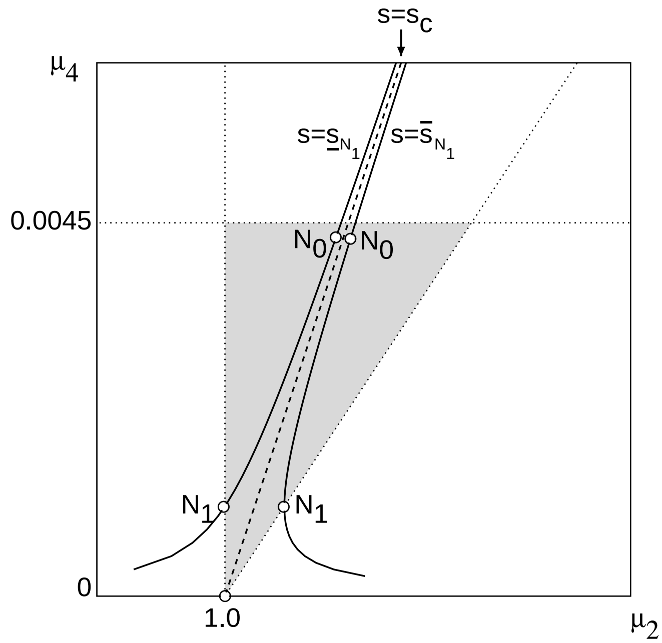

Figure 1. A schematic view of trajectories on (μ_2, μ_4)-plane in Theorem 2.1. Trajectories for $s = \overline{s}_{N_1}$ and for $s = \underline{s}_{N_1}$ (solid lines) and the critical trajectory for $s = s_c$ (broken line) are shown. The Gaussian fixed point corresponds to the point $(1.0, 0)$. The region defined by inequalities for (μ_2, μ_4) analogous to (2.13) and (2.14) (and (2.17)) is shaded.

The following theorem states our result in the strong coupling regime and is proved in Section 5.

Theorem 2.2 *The assumptions of Theorem 2.1 are satisfied for* $N_0 = 70$ *and* $N_1 = 100$, *where* $\underline{s}_{N_1}$ *and* $\overline{s}_{N_1}$ *satisfy*

$$1.7925671170092624 \leq \underline{s}_{N_1}, \quad \overline{s}_{N_1} \leq 1.7925671170092625 \ .$$

Proof of *Theorem 1.1 for* $d = 4$ *assuming Theorem 2.1 and Theorem 2.2.*
Theorem 2.1 and Theorem 2.2 imply that there exists $s_c \in [\underline{s}_{N_1}, \overline{s}_{N_1}]$ such that, for $s = s_c$, $\lim_{N\to\infty} \mu_{4,N} = 0$ and $\lim_{N\to\infty} \mu_{2,N} = 1$ hold. Then (2.6), (2.7), and (2.8) imply

$$\lim_{N\to\infty} \hat{h}_N(\xi) = e^{-\xi^2},$$

uniformly in ξ on any closed interval in $\mathbb{R}$. It is easy to see that $e^{-\xi^2}$ is the characteristic function of the massless Gaussian measure h_G, hence Theorem 1.1 holds for $d = 4$.

The bounds on $\underline{s}_{N_1}$ and $\overline{s}_{N_1}$ in Theorem 2.2 imply

$$1.7925671170092624 \leq s_c \leq 1.7925671170092625.$$

□

3 Truncated correlations.

In this section, we prepare basic (recursive) bounds on the truncated correlations that will be used in Section 4. The renormalization group transformation is decomposed as (2.3). Since the mapping $\mathcal{S}$ is simple, the essential part of our work is an analysis of $\mathcal{T}$. The consequence in this section is Proposition 3.1.

3.1 Recursions.

Note first that in terms of V_N the mapping $\mathcal{S}$ can be expressed as

$$\left(\mathcal{S}e^{-V_N}\right)(\xi) = e^{-2V_N\left(\frac{\sqrt{c}}{2}\xi\right)}. \tag{3.1}$$

Using (2.7), (2.10), (1.4) we also have

$$2V_N\left(\frac{\sqrt{c}}{2}\xi\right) = \sum_{n=1}^{\infty} 2^{1-(1+2/d)n}\mu_{2n,N}\xi^{2n}. \tag{3.2}$$

Next, write (2.5) as

$$\mathcal{T}g = \text{const.}\, g_{\beta/2}\,, \quad g_t = \exp(-t\triangle)g, \tag{3.3}$$

where $\triangle g(\xi) = \dfrac{d^2 g}{d\xi^2}(\xi)$, and $\beta = \frac{1}{2}(\sqrt{2}-1)$ for $d=4$. g_t is a solution to

$$\frac{\partial g_t}{\partial t} = -\triangle g_t\,, \quad g_0 = g.$$

Hence, if we put

$$g_t(\xi) = \exp(-V_t(\xi)),$$

then V_t satisfies

$$\frac{d}{dt}V_t = (\nabla V_t)^2 - \triangle V_t\,, \tag{3.4}$$

where $\nabla V_t(\xi) = \dfrac{\partial V_t}{\partial \xi}(\xi)$. In other words, V_{N+1} is given as a solution of (3.4) at $t = \beta/2$ (modulo constant term), with the initial condition (3.2) at $t=0$.

If we write

$$V_t(\xi) = \sum_{n=0}^{\infty} \mu_{2n}(t)\xi^{2n},$$

then (3.4) implies

$$\frac{d}{dt}\mu_{2n}(t) = -(2n+2)(2n+1)\mu_{2n+2}(t) + \sum_{\ell=1}^{n}(2\ell)(2n-2\ell+2)\mu_{2\ell}(t)\,\mu_{2n-2\ell+2}(t). \tag{3.5}$$

In particular, we have

$$\begin{aligned}
\frac{d}{dt}\mu_2(t) &= 4\mu_2(t)^2 - 12\mu_4(t), & (3.6)\\
\frac{d}{dt}\mu_4(t) &= 16\mu_2(t)\mu_4(t) - 30\mu_6(t), & (3.7)\\
\frac{d}{dt}\mu_6(t) &= 24\mu_2(t)\mu_6(t) + 16\mu_4(t)^2 - 56\mu_8(t), & (3.8)\\
\frac{d}{dt}\mu_8(t) &= 32\mu_2(t)\mu_8(t) + 48\mu_4(t)\mu_6(t) - 90\mu_{10}(t). & (3.9)
\end{aligned}$$

Thus, $\mu_{2n,N}$ and $\mu_{2n,N+1}$ are related for $d=4$ by e.g.,

$$\mu_2(0) = \frac{1}{\sqrt{2}}\mu_{2,N}\,,\ \mu_4(0) = \frac{1}{4}\mu_{4,N}\,,\ \mu_6(0) = \frac{1}{8\sqrt{2}}\mu_{6,N}\,,\ \mu_8(0) = \frac{1}{32}\mu_{8,N}\,,$$
$$\mu_{2,N+1} = \mu_2(\frac{\beta}{2}),\ \mu_{4,N+1} = \mu_4(\frac{\beta}{2}),\ \mu_{6,N+1} = \mu_6(\frac{\beta}{2}),\ \mu_{8,N+1} = \mu_8(\frac{\beta}{2}).$$

3.2 Bounds.

We first note that the quantities $\mu_n(t)$ obey Newman's inequalities: by comparing (2.5) and (3.3) we see that the correspondence $V_N \mapsto V(t)$ is obtained by a replacement $\beta \mapsto 2t$ in (1.2). Therefore $\mu_n(t)$ also is a truncated n point correlation of a measure to which arguments in [15] apply, hence an analogue of (2.8) holds:

$$0 \le \mu_{2n}(t) \le \frac{1}{n}(2\mu_4(t))^{n/2}, \ n = 3, 4, 5, \cdots . \tag{3.10}$$

We have to show decay of $\mu_{4,N}$ as $N \to \infty$. In case $d > 4$, the decay follows from (3.6) and (3.7) with d-dependent coefficients, namely, if we throw out the negative contributions $-\mu_4(t)$ and $-\mu_6(t)$ to the right hand sides of (3.6) and (3.7), respectively, then we have upper bounds on $\mu_2(t)$ and $\mu_4(t)$. This argument eventually yields exponential decay of $\mu_{4,N}$.

In case $d = 4$, the situation is more subtle, since the decay of $\mu_{4,N}$ is weak, i.e., powerlike instead of exponential. In order to derive the delicate bound on $\mu_4(t)$, a lower bound for $\mu_6(t)$ must be incorporated, which in turn needs an upper bound on $\mu_8(t)$. Thus, we have to deal with the equations (3.6)–(3.9). This is the principle of our estimation.

The result is the following:

Proposition 3.1 *Let $d = 4$ and N be a positive integer, and put*

$$r_N = \frac{1}{1-(\sqrt{2}-1)(\mu_{2,N}-1)} = \frac{1}{\sqrt{2}-(\sqrt{2}-1)\mu_{2,N}}\,, \tag{3.11}$$

$$\zeta_N = \frac{\sqrt{2}r_N - 1}{\sqrt{2}\mu_{2,N}} = \frac{r_N}{\mu_{2,N}} - \frac{1}{\sqrt{2}\mu_{2,N}}\,. \tag{3.12}$$

(i) If

$$\mu_{2,N} < 2 + \sqrt{2}, \tag{3.13}$$

then

$$\mu_{2,N+1} \le r_N \mu_{2,N}\,, \tag{3.14}$$

$$\mu_{2,N+1} \ge r_N \mu_{2,N} - 3r_N^2 \zeta_N \mu_{4,N}\,. \tag{3.15}$$

(ii) If, furthermore,

$$\frac{\mu_{4,N}}{4} \ge \frac{15}{8\sqrt{2}}\zeta_N\mu_{6,N} + \frac{21}{4}\zeta_N^2\mu_{4,N}^2, \tag{3.16}$$

$$\frac{\mu_{6,N}}{8\sqrt{2}} + \frac{1}{2}\zeta_N\mu_{4,N}^2 \ge 24\zeta_N^3\mu_{4,N}^3 + \frac{123}{8\sqrt{2}}\zeta_N^2\mu_{4,N}\mu_{6,N} + \frac{7}{8}\zeta_N\mu_{8,N}\,, \tag{3.17}$$

$$\frac{3}{2}\zeta_N\mu_{4,N} \ge 12\zeta_N^3\mu_{4,N}^2 + \frac{45}{8\sqrt{2}}\zeta_N^2\mu_{6,N}\,, \tag{3.18}$$

then

$$\mu_{2,N+1} \le r_N\mu_{2,N} - 3r_N^2(\zeta_N\mu_{4,N} - 8\zeta_N^3\mu_{4,N}^2 - \frac{15}{4\sqrt{2}}\zeta_N^2\mu_{6,N}), \tag{3.19}$$

$$\mu_{4,N+1} \ge r_N^4(\mu_{4,N} - \frac{15}{2\sqrt{2}}\zeta_N\mu_{6,N} - 21\zeta_N^2\mu_{4,N}^2), \tag{3.20}$$

$$\begin{aligned}\mu_{4,N+1} \le\ & r_N^4(\mu_{4,N} - \frac{15}{2\sqrt{2}}\zeta_N\mu_{6,N} - 21\zeta_N^2\mu_{4,N}^2 \\ & + \frac{705}{2\sqrt{2}}\zeta_N^3\mu_{4,N}\mu_{6,N} + 447\zeta_N^4\mu_{4,N}^3 + \frac{105}{4}\zeta_N^2\mu_{8,N}),\end{aligned} \tag{3.21}$$

$$\mu_{6,N+1} \le r_N^6(\frac{\mu_{6,N}}{\sqrt{2}} + 4\zeta_N\mu_{4,N}^2), \tag{3.22}$$

$$\mu_{6,N+1} \ge r_N^6(\frac{\mu_{6,N}}{\sqrt{2}} + 4\zeta_N\mu_{4,N}^2 - 192\zeta_N^3\mu_{4,N}^3 - \frac{123}{\sqrt{2}}\zeta_N^2\mu_{4,N}\mu_{6,N} - 7\zeta_N\mu_{8,N}), \tag{3.23}$$

$$\mu_{8,N+1} \le r_N^8(\frac{\mu_{8,N}}{2} + \frac{12}{\sqrt{2}}\zeta_N\mu_{4,N}\mu_{6,N} + 24\zeta_N^2\mu_{4,N}^3). \tag{3.24}$$

The rest of this section is devoted to a proof of Proposition 3.1.

Proof. Now, observe that $\bar{\mu_2}(t)$ defined by

$$\frac{d}{dt}\bar{\mu_2}(t) = 4\bar{\mu_2}(t)^2, \quad \bar{\mu_2}(0) = \frac{1}{\sqrt{2}}\mu_{2,N}\,, \tag{3.25}$$

is an upper bound of $\mu_2(t)$:

$$\mu_2(t) \le \bar{\mu_2}(t) = \frac{\mu_{2,N}}{\sqrt{2}} \frac{1}{1-2\sqrt{2}\mu_{2,N}t}\,. \tag{3.26}$$

This, at $t = \frac{\beta}{2}$ $(= \frac{\sqrt{2}-1}{4}$ for $d = 4)$ implies (3.14).

Put

$$\begin{aligned} M(t) &= \frac{1}{1-2\sqrt{2}\mu_{2,N}t}\,, \\ m(t) &= \bar{\mu_2}(t) - \mu_2(t). \end{aligned}$$

We have $m(t) \ge 0$, and (3.13) implies that $M(t)$ is increasing in $t \in [0, \beta/2]$.

By a change of variable $z = M(t) - 1$ $(dz = 2\sqrt{2}\mu_{2,N}M(t)^2 dt)$ and by putting

$$\hat{m}(z) = m(t)/M(t)^2, \quad \hat{\mu_4}(z) = \mu_4(t)/M(t)^4, \quad \hat{\mu_6}(z) = \mu_6(t)/M(t)^6, \quad \hat{\mu_8}(z) = \mu_8(t)/M(t)^8,$$

we have, from (3.6) – (3.9),

$$\hat{\mu_4}(z) = \frac{\mu_{4,N}}{4} + \frac{1}{\sqrt{2}\mu_{2,N}} \int_0^z (-8\hat{m}(z)\hat{\mu_4}(z) - 15\hat{\mu_6}(z))dz, \tag{3.27}$$

$$\hat{\mu_6}(z) = \frac{\mu_{6,N}}{8\sqrt{2}} + \frac{1}{\sqrt{2}\mu_{2,N}} \int_0^z (8\hat{\mu_4}(z)^2 - 12\hat{m}(z)\hat{\mu_6}(z) - 28\hat{\mu_8}(z))dz, \tag{3.28}$$

$$\hat{\mu_8}(z) = \frac{\mu_{8,N}}{32} + \frac{1}{\sqrt{2}\mu_{2,N}} \int_0^z (24\hat{\mu_4}(z)\hat{\mu_6}(z) - 16\hat{m}(z)\hat{\mu_8}(z) - 45\hat{\mu_{10}}(z))dz, \tag{3.29}$$

$$\hat{m}(z) = \frac{1}{\sqrt{2}\mu_{2,N}} \int_0^z (6\hat{\mu_4}(z) - 2\hat{m}(z)^2)dz, \tag{3.30}$$

The equations (3.27)–(3.30) with positivity of $\mu_{2n}(t)$ imply

$$\hat{\mu_4}(z) \le \frac{\mu_{4,N}}{4}, \tag{3.31}$$

$$\hat{\mu_6}(z) \le \frac{\mu_{6,N}}{8\sqrt{2}} + \frac{1}{\sqrt{2}\mu_{2,N}} \int_0^z 8\hat{\mu_4}(z)^2 dz \le \frac{\mu_{6,N}}{8\sqrt{2}} + \frac{\mu_{4,N}^2}{2\sqrt{2}\mu_{2,N}}z, \tag{3.32}$$

$$\hat{\mu_8}(z) \le \frac{\mu_{8,N}}{32} + \frac{1}{\sqrt{2}\mu_{2,N}} \int_0^z 24\hat{\mu_4}(z)\hat{\mu_6}(z)dz \le \frac{\mu_{8,N}}{32} + \frac{3}{8}\frac{\mu_{4,N}\mu_{6,N}}{\mu_{2,N}}z + \frac{3}{4}\frac{\mu_{4,N}^3}{\mu_{2,N}^2}z^2, \tag{3.33}$$

$$\hat{m}(z) \le \frac{1}{\sqrt{2}\mu_{2,N}} \int_0^z 6\hat{\mu_4}(z)dz \le \frac{3\mu_{4,N}}{2\sqrt{2}\mu_{2,N}}z. \tag{3.34}$$

In particular, (3.34) at $t = \frac{\beta}{2}$ $(z = M(\frac{\beta}{2}) - 1 = \sqrt{2}r_n - 1$ for $d = 4)$ implies (3.15).

Using (3.31), (3.32), (3.34) in (3.27), we have

$$\hat{\mu_4}(z) \ge \frac{\mu_{4,N}}{4} - \frac{15\mu_{6,N}}{16\mu_{2,N}}z - \frac{21\mu_{4,N}^2}{8\mu_{2,N}^2}z^2. \tag{3.35}$$

Using (3.32), (3.33), (3.34), (3.35) in (3.28) and (3.30) we further have

$$\hat{\mu_6}(z) \ge \frac{\mu_{6,N}}{8\sqrt{2}} + \frac{\mu_{4,N}^2}{2\sqrt{2}\mu_{2,N}}z - \frac{12\mu_{4,N}^3}{\sqrt{2}\mu_{2,N}^3}z^3 - \frac{123\mu_{4,N}\mu_{6,N}}{16\sqrt{2}\mu_{2,N}^2}z^2 - \frac{7\mu_{8,N}}{8\sqrt{2}\mu_{2,N}}z, \tag{3.36}$$

$$\hat{m}(z) \ge \frac{3\mu_{4,N}}{2\sqrt{2}\mu_{2,N}}z - \frac{6\mu_{4,N}^2}{\sqrt{2}\mu_{2,N}^3}z^3 - \frac{45\mu_{6,N}}{16\sqrt{2}\mu_{2,N}^2}z^2. \tag{3.37}$$

When $d=4$, $\beta=\dfrac{\sqrt{2}-1}{2}$ and $z=M(\dfrac{\beta}{2})-1=\sqrt{2}r_N-1$ $(M(\dfrac{\beta}{2})=\sqrt{2}r_N)$. Then the assumptions (3.16) – (3.18) of Proposition 3.1 imply that the right hand sides of (3.35), (3.36), and (3.37) are non-negative at $t=\dfrac{\beta}{2}$. On the other hand, they are concave in z for $z\geq 0$. Recall also that $z=M(t)-1$ is increasing in $t\in[0,\beta/2]$. Therefore, they are non-negative for all $t\in[0,\beta/2]$. Using (3.35), (3.36), and (3.37) in (3.27), we therefore have

$$\begin{aligned}
\hat{\mu_4}(z) \;\leq\; & \frac{\mu_{4,N}}{4}-\frac{1}{\sqrt{2}\mu_{2,N}}\times \\
& \times\int_0^z\left(8\left(\frac{3\mu_{4,N}}{2\sqrt{2}\mu_{2,N}}z-\frac{6\mu_{4,N}^2}{\sqrt{2}\mu_{2,N}^3}z^3-\frac{45\mu_{6,N}}{16\sqrt{2}\mu_{2,N}^2}z^2\right)\left(\frac{\mu_{4,N}}{4}-\frac{15\mu_{6,N}}{16\mu_{2,N}}z-\frac{21\mu_{4,N}^2}{8\mu_{2,N}^2}z^2\right)\right. \\
& \left.+15\left(\frac{\mu_{6,N}}{8\sqrt{2}}+\frac{\mu_{4,N}^2}{2\sqrt{2}\mu_{2,N}}z-\frac{12\mu_{4,N}^3}{\sqrt{2}\mu_{2,N}^3}z^3-\frac{123\mu_{4,N}\mu_{6,N}}{16\sqrt{2}\mu_{2,N}^2}z^2-\frac{7\mu_{8,N}}{8\sqrt{2}\mu_{2,N}}z\right)\right)dz \\
\leq\; & \frac{\mu_{4,N}}{4}-\frac{15\mu_{6,N}}{16\mu_{2,N}}z-\frac{21\mu_{4,N}^2}{8\mu_{2,N}^2}z^2+\frac{705\mu_{4,N}\mu_{6,N}}{32\mu_{2,N}^3}z^3+\frac{447\mu_{4,N}^3}{16\mu_{2,N}^4}z^4+\frac{105\mu_{8,N}}{32\mu_{2,N}^2}z^2. \qquad (3.38)
\end{aligned}$$

Recalling that at $t=\beta/2$ $(z=M(\frac{\beta}{2})-1=\sqrt{2}r_N-1)$ we have

$$\begin{aligned}
\bar{\mu_2}(\frac{\beta}{2}) &= r_N\mu_{2,N}\,, \\
\mu_{2,N+1} &= r_N\mu_{2,N}-\hat{m}(\sqrt{2}r_N-1)M(\frac{\beta}{2})^2, \\
\mu_{4,N+1} &= \hat{\mu_4}(\sqrt{2}r_N-1)M(\frac{\beta}{2})^4, \\
\mu_{6,N+1} &= \hat{\mu_6}(\sqrt{2}r_N-1)M(\frac{\beta}{2})^6, \\
\mu_{8,N+1} &= \hat{\mu_8}(\sqrt{2}r_N-1)M(\frac{\beta}{2})^8,
\end{aligned}$$

we see that (3.37), (3.35), (3.38), (3.32), (3.36), (3.33) imply (3.19) – (3.24), respectively.

This completes a proof of Proposition 3.1. □

4 Bleher–Sinai argument.

In order to show Theorem 2.1, we confirm existence of a critical parameter $s=s_c$ by means of Bleher-Sinai argument, and, at the same time, we derive the expected decay of $\mu_{4,N}$. In Bleher-Sinai argument, monotonicity of $\underline{s}_N$ and $\overline{s}_N$ with respect to N is essential.

Proposition 4.1 *Let $d=4$. Then the following hold.*

(1) If $\mu_{2,N}-1<0$ then $\mu_{2,N+1}<\mu_{2,N}$.

(2) If $\dfrac{1}{4}>\mu_{2,N}-1\geq\dfrac{3}{\sqrt{2}}\mu_{4,N}$ then $\mu_{2,N+1}\geq\mu_{2,N}$.

Proof. Note that for both cases in the statement, the assumption (3.13) in Proposition 3.1 holds. Hence, (3.14), with (3.11) and monotonicity of $\mu_{2,N}$, implies

$$\mu_{2,N}-1<0\implies r_N<1\implies\mu_{2,N+1}<\mu_{2,N}\,. \qquad (4.1)$$

Next we see that (3.15), with (3.11) and (3.12), implies

$$\frac{\mu_{2,N}-1}{\mu_{4,N}}\geq\frac{3r_N(\sqrt{2}r_N-1)}{(2-\sqrt{2})\mu_{2,N}^2}\implies\mu_{2,N+1}\geq\mu_{2,N}\,. \qquad (4.2)$$

Put

$$L_1(x) = \frac{3}{\sqrt{2}x(\sqrt{2}-(\sqrt{2}-1)x)^2}\,.$$

Then by straightforward calculation we see

$$1 \le x \le \frac{5}{4} \implies L_1(x) \le L_1(1) = \frac{3}{\sqrt{2}}\,,$$

and (3.11) implies

$$L_1(\mu_{2,N}) = \frac{3r_N(\sqrt{2}r_N - 1)}{(2-\sqrt{2})\mu_{2,N}^2}\,.$$

Therefore (4.2) implies that

$$\frac{1}{4} > \mu_{2,N} - 1 \ge \frac{3}{\sqrt{2}}\mu_{4,N} \implies \mu_{2,N+1} \ge \mu_{2,N}\,. \tag{4.3}$$

□

Corollary 4.2 *Let $d=4$. Then, for the $\underline{s}_N$ defined in (2.11), it holds that $\underline{s}_N \le \underline{s}_{N+1}$.*

Proof. Since $\mu_{2,N}$ is increaisng in s, if $s < \underline{s}_N$ then $\mu_{2,N} < 1$, hence Proposition 4.1 implies $\mu_{2,N+1} < \mu_{2,N} < 1$, further implying $s < \underline{s}_{N+1}$. Hence the statement holds. □

For later convenience, define

$$r_N^* = \frac{1}{1-(\sqrt{2}-1)\dfrac{3}{\sqrt{2}}\mu_{4,N}}\,, \tag{4.4}$$

$$\zeta_{*N} = 1 - \frac{1}{\sqrt{2}}\,, \tag{4.5}$$

$$\zeta_N^* = \frac{\sqrt{2}r_N^* - 1}{\sqrt{2}(1+\dfrac{3}{\sqrt{2}}\mu_{4,N})}\,, \tag{4.6}$$

Then we see that if (2.13) holds, then we have, from (3.11) and (3.12),

$$1 < r_N < r_N^*, \tag{4.7}$$

$$\zeta_{*N} < \zeta_N < \zeta_N^*. \tag{4.8}$$

Proposition 4.3 *Let $d=4$ and put*

$$\alpha_0 = 0.0045, \quad \alpha_1 = 1.6, \quad \alpha_2 = 6.07, \quad \alpha_3 = 48.469\,.$$

Assume that there exists an integer N such that (3.13) and

$$(0 \le)\quad \mu_{4,N} \le \alpha_0, \tag{4.9}$$

$$\alpha_1\mu_{4,N}^2 \le \mu_{6,N} \le \alpha_2\mu_{4,N}^2, \tag{4.10}$$

$$(0 \le)\quad \mu_{8,N} \le \alpha_3\mu_{4,N}^3, \tag{4.11}$$

hold. Then (3.16)–(3.18) hold, and the following also hold:

$$(0 \le)\quad \mu_{4,N+1} \le \mu_{4,N}(1-0.08\mu_{4,N}) \quad (\le \alpha_0), \tag{4.12}$$

$$\alpha_1\mu_{4,N+1}^2 \le \mu_{6,N+1} \le \alpha_2\mu_{4,N+1}^2, \tag{4.13}$$

$$(0 \le)\quad \mu_{8,N+1} \le \alpha_3\mu_{4,N+1}^3. \tag{4.14}$$

Proof. For $x \geq 0$ put

$$\begin{aligned}
\ell_r(x) &= \frac{1}{1-(\sqrt{2}-1)\dfrac{3}{\sqrt{2}}\,x}\,, \\
\ell_d(x) &= 1-\frac{1}{\sqrt{2}}\,, \\
\ell_u(x) &= \frac{\sqrt{2}\ell_r(x)-1}{\sqrt{2}(1+\dfrac{3}{\sqrt{2}}\,x)}\,, \\
L_2(x) &= 1-(\frac{15}{2\sqrt{2}}\alpha_2\ell_u(x)+21\ell_u(x)^2)\,x. \qquad (4.15)
\end{aligned}$$

In particular, (4.4), (4.5), (4.6) imply

$$r_N^* = \ell_r(\mu_{4,N}), \quad \zeta_{*N} = \ell_d(\mu_{4,N}), \quad \zeta_N^* = \ell_u(\mu_{4,N}).$$

By explicit calculation, we see that

$$L_2(x) > 0, \quad 0 \leq x \leq \alpha_0\,. \qquad (4.16)$$

The right hand side of (3.16) is then bounded from above by

$$\frac{1}{4}\mu_{4,N}(1-L_2(\mu_{4,N})) \leq \frac{1}{4}\mu_{4,N}\,,$$

hence (3.16) holds. Similarly, (3.18) is seen to hold for $0 \leq \mu_{4,N} \leq \alpha_0$, if we note that the right hand side of (3.18) is bounded from above by

$$\zeta_N\mu_{4,N}^2(12\zeta_N^* + \frac{45}{8\sqrt{2}}\zeta_N^*\alpha_2) \leq \frac{3}{4}(1-L_2(\mu_{4,N}))\mu_{4,N} \leq \frac{3}{2}\mu_{4,N}\,.$$

The condition (3.17) is seen to hold with similar argument, if we note the right hand side is bounded from above by

$$\zeta_N\mu_{4,N}^3(24\zeta_N^2 + \frac{123}{8\sqrt{2}}\zeta_N\alpha_2 + \frac{7}{8}\alpha_3),$$

while the left hand side is bounded from below by

$$\mu_{4,N}^2(\frac{\alpha_1}{8\sqrt{2}} + \frac{1}{2}\zeta_N).$$

Therefore, the conclusions of Proposition 3.1 hold, in particular, (3.20)–(3.24) imply

$$\begin{aligned}
\mu_{4,N+1} &\geq r_N^4\mu_{4,N}\left(1-(\frac{15}{2\sqrt{2}}\zeta_N\alpha_2+21\zeta_N^2)\mu_{4,N}\right), && (4.17) \\
\frac{\mu_{4,N+1}}{\mu_{4,N}} &\leq r_N^4\left(1-(\frac{15}{2\sqrt{2}}\zeta_N\alpha_1+21\zeta_N^2)\mu_{4,N}+(\frac{705}{2\sqrt{2}}\zeta_N^3\alpha_2+447\zeta_N^4+\frac{105}{4}\zeta_N^2\alpha_3)\mu_{4,N}^2\right), && (4.18) \\
\frac{\mu_{6,N+1}}{\mu_{4,N+1}^2} &\leq \left(\frac{\mu_{4,N}}{\mu_{4,N+1}}\right)^2 r_N^6\left(\frac{\alpha_2}{\sqrt{2}}+4\zeta_N\right), && (4.19) \\
\frac{\mu_{6,N+1}}{\mu_{4,N+1}^2} &\geq \left(\frac{\mu_{4,N}}{\mu_{4,N+1}}\right)^2 r_N^6\left(\frac{\alpha_1}{\sqrt{2}}+4\zeta_N-(192\zeta_N^3+\frac{123}{\sqrt{2}}\zeta_N^2\alpha_2+7\zeta_N\alpha_3)\mu_{4,N}\right), && (4.20) \\
\frac{\mu_{8,N+1}}{\mu_{4,N+1}^3} &\leq \left(\frac{\mu_{4,N}}{\mu_{4,N+1}}\right)^3 r_N^8\left(\frac{\alpha_3}{2}+\frac{12}{\sqrt{2}}\zeta_N\alpha_2+24\zeta_N^2\right). && (4.21)
\end{aligned}$$

Rewriting (4.17), using (4.7) and (4.8), we have

$$\frac{\mu_{4,N}}{\mu_{4,N+1}} \leq \frac{1}{r_N^4}\frac{1}{1-(\dfrac{15}{2\sqrt{2}}\zeta_N\alpha_2+21\zeta_N^2)\mu_{4,N}} \leq \frac{1}{L_2(\mu_{4,N})}\,. \qquad (4.22)$$

This and (4.19) imply

$$\frac{\mu_{6,N+1}}{\mu_{4,N+1}^2} \le \frac{\frac{1}{\sqrt{2}}\alpha_2 + 4\ell_u(\mu_{4,N})}{L_2(\mu_{4,N})^2}\,.$$

By explicit calculation, we see that

$$0 \le x \le \alpha_0 \implies \frac{\frac{1}{\sqrt{2}}\alpha_2 + 4\ell_u(x)}{L_2(x)^2} \le \alpha_2.$$

Therefore the upper bound in (4.13) holds.

In a similar way, we note that (4.21) and (4.22) imply

$$\frac{\mu_{8,N+1}}{\mu_{4,N+1}^3} \le \frac{\frac{1}{2}\alpha_3 + \frac{12}{\sqrt{2}}\ell_u(\mu_{4,N})\alpha_2 + 24\ell_u(\mu_{4,N})^2}{L_2(\mu_{4,N})^3}\,.$$

By explicit calculation, we see that

$$0 \le x \le \alpha_0 \implies \frac{\frac{1}{2}\alpha_3 + \frac{12}{\sqrt{2}}\ell_u(x)\alpha_2 + 24\ell_u(x)^2}{L_2(x)^2} \le \alpha_3.$$

Therefore (4.14) holds.

Similarly, from (4.20) and (4.18), we have

$$\begin{aligned}
&\frac{\mu_{6,N+1}}{\mu_{4,N+1}^2} \\
&\quad\ge \frac{1}{\ell_r(\mu_{4,N})^2} \\
&\times \frac{\frac{\alpha_1}{\sqrt{2}} + 4\ell_d(\mu_{4,N}) - (192\ell_u(\mu_{4,N})^3 + \frac{123}{\sqrt{2}}\ell_u(\mu_{4,N})^2\alpha_2 + 7\ell_u(\mu_{4,N})\alpha_3)\mu_{4,N}}{\left(1 - (\frac{15}{2\sqrt{2}}\ell_d(\mu_{4,N})\alpha_1 + 21\ell_d(\mu_{4,N})^2)\mu_{4,N} + (\frac{705}{2\sqrt{2}}\ell_u(\mu_{4,N})^3\alpha_2 + 447\ell_u(\mu_{4,N})^4 + \frac{105}{4}\ell_u(\mu_{4,N})^2\alpha_3)\mu_{4,N}^2\right)^2} \\
&\quad\ge \alpha_1\,,
\end{aligned}$$

if $0 \le \mu_{4,N} \le \alpha_0$. Therefore the lower bound in (4.13) holds.

Finally, from (4.18), we have, again with similar argument,

$$\begin{aligned}
\frac{\mu_{4,N+1}}{\mu_{4,N}} &\le \ell_r(\mu_{4,N})^4\Bigg(1 - (\frac{15}{2\sqrt{2}}\ell_d(\mu_{4,N})\alpha_1 + 21\ell_d(\mu_{4,N})^2)\mu_{4,N} \\
&\qquad + (\frac{705}{2\sqrt{2}}\ell_u(\mu_{4,N})^3\alpha_2 + 447\ell_u(\mu_{4,N})^4 + \frac{105}{4}\ell_u(\mu_{4,N})^2\alpha_3)\mu_{4,N}^2\Bigg) \\
&\le 1 - 0.08\mu_{4,N}\,,
\end{aligned}$$

if $0 \le \mu_{4,N} \le \alpha_0$. Therefore (4.14) holds. □

Corollary 4.4 *Let $d = 4$, and assume that for some N the assumptions (4.9) – (4.11) in Proposition 4.3 hold for all s satisfying $\underline{s}_N \le s \le \overline{s}_N$, where $\underline{s}_N$ and $\overline{s}_N$ are defined in (2.11) and (2.12). Then it holds that $\overline{s}_{N+1} \le \overline{s}_N$.*

Proof. By (4.9), $1 + \frac{3}{\sqrt{2}}\mu_{4,N} < 2 + \sqrt{2}$, if $\underline{s}_N \le s \le \overline{s}_N$. Hence, by (2.12),

$$\overline{s}_N = \inf\{s > 0 \mid \mu_{2,N} \ge 1 + \frac{3}{\sqrt{2}}\mu_{4,N}\},$$

and, from monotonicity of $\mu_{2,N}$ in s, (3.13) holds if $s \le \overline{s}_N$.

Continuity of $\mu_{2,N}$ and $\mu_{4,N}$ in s imply

$$\mu_{2,N} = 1 + \frac{3}{\sqrt{2}}\mu_{4,N}\,, \quad \text{if} \quad s = \overline{s}_n\,.$$

(In particular, we may assume that $\frac{5}{4} > \mu_{2,N}$.) Hence Proposition 4.1 implies

$$\mu_{2,N+1} \ge 1 + \frac{3}{\sqrt{2}}\mu_{4,N}\,, \quad \text{for } s = \overline{s}_N\,. \tag{4.23}$$

By assumptions at $s = \overline{s}_N$, we see, from Proposition 4.3, that $\mu_{4,N+1} \le \mu_{4,N}$, which, with (4.23), implies

$$\mu_{2,N+1} \ge 1 + \frac{3}{\sqrt{2}}\mu_{4,N+1}\,.$$

This proves $\overline{s}_{N+1} \le \overline{s}_N$. □

Proof of *Theorem 2.1.* Note first that Corollary 4.2 implies

$$\underline{s}_N \le \underline{s}_{N+1}\,, \quad N = N_1, N_1+1, N_1+2, \cdots\,. \tag{4.24}$$

With assumptions of the theorem and by induction on N, Proposition 4.3 implies that for any s satisfying $\underline{s}_{N_1} \le s \le \overline{s}_{N_1}$, the bounds (4.9) – (4.11) hold for $N = N_1$. Hence Corollary 4.4 implies $\overline{s}_{N_1+1} \le \overline{s}_{N_1}$.

Also since $s \le \overline{s}_{N_1}$ implies (3.13) for $N = N_1$, Proposition 4.3 implies that (4.9) – (4.11) hold for $N = N_1 + 1$ and $\underline{s}_{N_1+1} \le s \le \overline{s}_{N_1+1}$. We can proceed with induction on N and repeat this argument to conclude that (4.12) – (4.14) hold for $\underline{s}_N \le s \le \overline{s}_N$, $N = N_1, N_1+1, N_1+2, \cdots$, and

$$\overline{s}_{N+1} \le \overline{s}_N\,, \quad N = N_1, N_1+1, N_1+2, \cdots\,. \tag{4.25}$$

The bounds (4.24) and (4.25) imply that a sequence of closed intervals on $\mathbb{R}$

$$[\underline{s}_{N_1}, \overline{s}_{N_1}] \supset [\underline{s}_{N_1+1}, \overline{s}_{N_1+1}] \supset [\underline{s}_{N_1+2}, \overline{s}_{N_1+2}] \supset \cdots\,,$$

is contracting, hence there exists an s_c, satisfying $\underline{s}_{N_1} \le s_c \le \overline{s}_{N_1}$, such that

$$\underline{s}_N \le s_c \le \overline{s}_N\,, \quad N = N_1, N_1+1, N_1+2, \cdots\,.$$

Hence, in particular, (4.12) holds for all integer $N \ge N_1$ at $s = s_c$. This implies

$$\lim_{N\to\infty} \mu_{4,N} = 0\,,$$

at $s = s_c$.

Also we see that if $s = s_c$ then (2.13) holds for all $N \ge N_1$. Therefore we have

$$\lim_{N\to\infty} \mu_{2,N} = 1\,,$$

at $s = s_c$. This completes a proof of Theorem 2.1. □

5 Strong coupling problem.

We shall prove Theorem 2.2 by (computer-aided) brute force evaluation of the Taylor coefficients of $\hat{h}_N(\xi)$ instead of $V_N(\xi)$.

5.1 Taylor expansion.

Define the Taylor coefficients $a_{n,N}$, $n \in \mathbb{Z}_+$, of $\hat{h}_N$ by

$$\hat{h}_N(\xi) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} a_{n,N} \xi^{2n}. \tag{5.1}$$

In particular, $a_{0,N} = \hat{h}_N(0) = 1$. Note also that

$$a_{n,N} \geq 0, \ n \in \mathbb{Z}_+ .$$

$\mu_{n,N}$ and $a_{n,N}$ are related, e.g., as

$$\mu_{2,N} = a_{1,N} , \quad \mu_{4,N} = \frac{a_{1,N}^2 - a_{2,N}}{2} , \quad \mu_{6,N} = \frac{a_{1,N}^3}{3} - \frac{a_{1,N}\, a_{2,N}}{2} + \frac{a_{3,N}}{6} ,$$
$$\mu_{8,N} = \frac{a_{1,N}^4}{4} - \frac{a_{1,N}^2\, a_{2,N}}{2} + \frac{a_{2,N}^2}{8} + \frac{a_{1,N}\, a_{3,N}}{6} - \frac{a_{4,N}}{24} .$$

For Ising measure $h_0 = h_{I,s}$,

$$a_{n,0} = (-1)^n \frac{n!}{(2n)!} \frac{d^{2n} \hat{h}_0}{d\, \xi^{2n}}(0) = \frac{n!}{(2n)!} \int x^{2n}\, h_{I,s}(x) dx = \frac{n!}{(2n)!} s^{2n}, \quad n \in \mathbb{Z}_+ . \tag{5.2}$$

Note that one of the Newman inequalities (see (A.6)), or the Gaussian inequalities, imply that

$$a_{n,N} \leq a_{1,N}^n = \mu_{2,N}^n, \quad n \in \mathbb{Z}_+ . \tag{5.3}$$

Define $b_{n,N}$, $n \in \mathbb{Z}_+$, by

$$(\mathcal{S}\hat{h}_N)(\xi) = \hat{h}_N(\frac{\sqrt{c}}{2}\xi)^2 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} b_{n,N} \xi^{2n},$$

where $\mathcal{S}$ is in (2.4). Then

$$b_{n,N} = \left(\frac{c}{4}\right)^n \sum_{\ell=0}^{n} \binom{n}{\ell} a_{\ell,N}\, a_{n-\ell,N} , \quad n \in \mathbb{Z}_+ . \tag{5.4}$$

With (5.3) we have,

$$b_{n,N} \leq \left(\frac{c\mu_{2,N}}{2}\right)^n , \quad n \in \mathbb{Z}_+ . \tag{5.5}$$

Next define $\tilde{a}_{n,N}$, $n \in \mathbb{Z}_+$, by

$$\sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{\beta}{2}\right)^m \frac{d^{2m}}{d\xi^{2m}} \mathcal{S}\hat{h}_N\, (\xi) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \tilde{a}_{n,N} \xi^{2n}.$$

Then

$$\tilde{a}_{n,N} = \sum_{m=0}^{\infty} \left(\frac{\beta}{2}\right)^m b_{m+n,N} \frac{(2m+2n)! n!}{m!(m+n)!(2n)!} , \quad n \in \mathbb{Z}_+ , \tag{5.6}$$

and (2.5) implies

$$\hat{h}_{N+1}(\xi) = \frac{1}{\tilde{a}_{0,N}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \tilde{a}_{n,N} \xi^{2n},$$

where we fixed the constant in the definition of $\mathcal{T}$ by $\hat{h}_{N+1}(0) = 1$. Comparing this with (5.1) we obtain a recursion relation in N for $a_{n,N}$:

$$a_{n,N+1} = \frac{\tilde{a}_{n,N}}{\tilde{a}_{0,N}} , \quad n \in \mathbb{Z}_+, \ N \in \mathbb{Z}_+ . \tag{5.7}$$

5.2 Truncation.

We will evaluate a finite number, say M, of $a_{n,N}$'s ($n = 1, 2, \cdots, M$) explicitly with aid of computer calculations, by evaluating $a_{n,N}$, $n > M$, 'theoretically'. For this, we need to give bounds of series in (5.4) and (5.6) in terms of sums of finite terms. The following proposition serves for this purpose.

Proposition 5.1 *Let M be a positive integer, and define*

$$\underline{b}_{n,N}\,,\ \bar{b}_{n,N}\,,\quad n = 0, 1, 2, \cdots, 2M,$$

and

$$\underline{\tilde{a}}_{n,N}\,,\ \tilde{\bar{a}}_{n,N}\,,\ \underline{a}_{n,N}\,,\ \bar{a}_{n,N}\,,\quad n = 0, 1, 2, \cdots, M,$$

inductively in $N \in \mathbb{Z}_+$, by

$$\underline{a}_{n,0} = \bar{a}_{n,0} = \frac{n!}{(2n)!} s^{2n}, \quad n = 0, 1, 2, \cdots, M,$$

and

$$\underline{b}_{n,N} = \left(\frac{c}{4}\right)^n \times \begin{cases} \displaystyle\sum_{\ell=0}^{n} \binom{n}{\ell} \underline{a}_{\ell,N}\, \underline{a}_{n-\ell,N}\,, & 0 \le n \le M, \\ \displaystyle\sum_{\ell=n-M}^{M} \binom{n}{\ell} \underline{a}_{\ell,N}\, \underline{a}_{n-\ell,N}\,, & M < n \le 2M, \end{cases} \tag{5.8}$$

$$\bar{b}_{n,N} = \begin{cases} \displaystyle\left(\frac{c}{4}\right)^n \sum_{\ell=0}^{n} \binom{n}{\ell} \bar{a}_{\ell,N}\, \bar{a}_{n-\ell,N}\,, & 0 \le n \le M, \\ \displaystyle\min\left\{ \left(\frac{c}{4}\right)^n \sum_{n-M \le \ell \le M} \binom{n}{\ell} \bar{a}_{\ell,N}\, \bar{a}_{n-\ell,N} + \overline{\Delta b}_{n,N},\ \left(\frac{c\bar{a}_{1,N}}{2}\right)^n \right\}, & M < n \le 2M, \end{cases} \tag{5.9}$$

$$\underline{\tilde{a}}_{n,N} = \sum_{m=0}^{2M-n} \left(\frac{\beta}{2}\right)^m \underline{b}_{m+n,N} \frac{(2m+2n)!n!}{m!(m+n)!(2n)!}\,, \quad 0 \le n \le M, \tag{5.10}$$

$$\tilde{\bar{a}}_{n,N} = \sum_{m=0}^{2M-n} \left(\frac{\beta}{2}\right)^m \bar{b}_{m+n,N} \frac{(2m+2n)!n!}{m!(m+n)!(2n)!} + \overline{\Delta a}_{n,N}\,, \quad 0 \le n \le M, \tag{5.11}$$

$$\underline{a}_{n,N+1} = \frac{\underline{\tilde{a}}_{n,N}}{\tilde{\bar{a}}_{0,N}}, \quad \bar{a}_{n,N+1} = \frac{\tilde{\bar{a}}_{n,N}}{\underline{\tilde{a}}_{0,N}}, \quad 1 \le n \le M, \tag{5.12}$$

and

$$\underline{a}_{0,N+1} = \bar{a}_{0,N+1} = 1,$$

where we put

$$\overline{\Delta b}_{n,N} = 2\left(\frac{c\,\bar{a}_{1,N}}{4}\right)^n \binom{n}{n-M-1} \times \frac{1}{1 - \frac{n-M}{M+1} e^{-1/(M+1)}} \times \frac{\bar{a}_{M,N}}{\underline{a}_{1,N}^{\,M}}\,, \tag{5.13}$$

and

$$\overline{\Delta a}_{n,N} = \left(\frac{1}{2\beta}\right)^n \frac{(\beta c\bar{a}_{1,N})^{2M+1}}{1 - 2\beta c\bar{a}_{1,N}} \binom{N}{n} \times \frac{\bar{a}_{M,N}}{\underline{a}_{1,N}^{\,M}}\,. \tag{5.14}$$

If for an integer N_1 it holds that

$$\bar{a}_{1,N} < \frac{1}{2\beta c}\,, \quad 0 \le N \le N_1\,, \tag{5.15}$$

then $a_{n,N}$, $b_{n,N}$, $\tilde{a}_{n,N}$, $n \in \mathbb{Z}_+$, $N \in \mathbb{Z}_+$, defined inductively by (5.2), (5.4), (5.6), (5.7), satisfy, for all $N \le N_1$,

$$\begin{aligned}
&\underline{b}_{n,N} \le b_{n,N} \le \bar{b}_{n,N}\,, \quad n = 0,1,2,\cdots,2M,\\
&\underline{\tilde{a}}_{\,n,N} \le \tilde{a}_{n,N} \le \bar{\tilde{a}}_{n,N}\,, \quad n = 0,1,2,\cdots,M,\\
&\underline{a}_{\,n,N} \le a_{n,N} \le \bar{a}_{n,N}\,, \quad n = 0,1,2,\cdots,M.
\end{aligned} \tag{5.16}$$

The rest of this subsection is devoted to a proof of this proposition.

Proof. The claimed bounds on $a_{n,N}$ in (5.16) hold for $N = 0$. We proceed by induction on N, and assume that they hold for N.

By comparing (5.4) with (5.8), and noting that $a_{n,N}$ are non-negative, we see that the lower bound for $b_{n,N}$ in (5.16) holds.

Assume for a moment that the upper bound for $b_{n,N}$ in (5.16) also holds. Then comparing (5.6) with (5.10), we see that the lower bound for $\tilde{a}_{n,N}$ in (5.16) holds. If the upper bound for $\tilde{a}_{n,N}$ also holds, then (5.7) and (5.12) imply that the bounds for $a_{n,N+1}$ in (5.16) also hold.

Hence we are left with proving the upper bounds for $b_{n,N}$ and $\tilde{a}_{n,N}$ in (5.16).

Upper bound on $b_{n,N}$. Note first that if $n \le M$, then

$$b_{n,N} = \left(\frac{c}{4}\right)^n \sum_{\ell=0}^{n} \binom{n}{\ell} a_{\ell,N}\, a_{n-\ell,N} \le \left(\frac{c}{4}\right)^n \sum_{\ell=0}^{n} \binom{n}{\ell} \bar{a}_{\ell,N}\, \bar{a}_{n-\ell,N} = \bar{b}_{n,N}\,,$$

hence $b_{n,N} \le \bar{b}_{n,N}$ holds. Also, (5.5) implies

$$b_{n,N} \le \left(\frac{c\mu_{2,N}}{2}\right)^n \le \left(\frac{c\bar{a}_{1,N}}{2}\right)^n,$$

hence it suffices to prove

$$b_{n,N} \le \left(\frac{c}{4}\right)^n \sum_{n-M \le \ell \le M} \binom{n}{\ell} \bar{a}_{\ell,N}\, \bar{a}_{n-\ell,N} + \overline{\Delta\bar{b}}_{n,N}, \quad M < n \le 2M. \tag{5.17}$$

To prove (5.17), first note

$$\begin{aligned}
\Delta\bar{b}_{n,N} &= b_{n,N} - \left(\frac{c}{4}\right)^n \sum_{n-M \le \ell \le M} \binom{n}{\ell} \bar{a}_{\ell,N}\, \bar{a}_{n-\ell,N}\\
&\le \left(\frac{c}{4}\right)^n \sum_{0 \le \ell < n-M \text{ or } M < \ell \le n} \binom{n}{\ell} a_{\ell,N}\, a_{n-\ell,N}\,.
\end{aligned} \tag{5.18}$$

Using the Newman inequalities (A.6) we see that if $\ell > M$

$$a_{\ell,N} \le a_{M,N} a_{\ell-M,N} \le a_{M,N}\, a_{1,N}^{\ell-M}. \tag{5.19}$$

Hence

$$\begin{aligned}
\Delta\bar{b}_{n,N} &\le \left(\frac{c}{4}\right)^n \left(\sum_{0 \le \ell < n-M} \binom{n}{\ell} a_{\ell,N}\, a_{M,N}\, a_{1,N}^{n-\ell-M} + \sum_{M < \ell \le n} \binom{n}{\ell} a_{M,N}\, a_{1,N}^{\ell-M} a_{n-\ell,N} \right)\\
&\le 2\left(\frac{c\, a_{1,N}}{4}\right)^n \frac{a_{M,N}}{a_{1,N}^{M}} \sum_{\ell=0}^{n-M-1} \binom{n}{\ell},
\end{aligned} \tag{5.20}$$

where we also used (5.5). Write the summation in the right hand side as

$$\sum_{\ell=0}^{n-M-1}\binom{n}{\ell} = \binom{n}{n-M-1}\left[1+\frac{n-M-1}{M+2}+\frac{n-M-1}{M+2}\frac{n-M-2}{M+3}\right.$$
$$\left.+\frac{n-M-1}{M+2}\frac{n-M-2}{M+3}\frac{n-M-3}{M+4}+\cdots\right]. \qquad (5.21)$$

Noting that

$$\frac{a-x}{1+x}\le ae^{-2x},\quad a\in(0,1],\ x\in[0,1], \qquad (5.22)$$

we find, by putting $a=\dfrac{n-M}{M+1}$ and $\epsilon=\dfrac{1}{M+1}$,

$$\frac{n-M-k}{M+k+1}=\frac{a-k\epsilon}{1+k\epsilon}\le a\,e^{-2k\epsilon}. \qquad (5.23)$$

Hence (5.21) has a bound

$$\sum_{\ell=0}^{n-M-1}\binom{n}{\ell}\le\binom{n}{n-M-1}\times\sum_{k=0}^{\infty}a^k\,e^{-k(k+1)\epsilon}\le\binom{n}{n-M-1}\times\frac{1}{1-ae^{-\epsilon}},\quad a=\frac{n-M}{M+1},\ \epsilon=\frac{1}{M+1},$$

which implies

$$\Delta\bar{b}_{n,N}\le\overline{\Delta\bar{b}}_{n,N}, \qquad (5.24)$$

where $\overline{\Delta\bar{b}}_{n,N}$ is defined in (5.13). This proves (5.17).

Upper bound on $\tilde{a}_{n,N}$. Put

$$\begin{aligned}\Delta\bar{a}_{\ell,N} &= \tilde{a}_{\ell,N}-\sum_{m=0}^{2M-\ell}\left(\frac{\beta}{2}\right)^m\bar{b}_{m+\ell,N}\frac{(2m+2\ell)!\ell!}{m!(m+\ell)!(2\ell)!}\\ &\le \sum_{m=2M+1-\ell}^{\infty}\left(\frac{\beta}{2}\right)^m b_{m+\ell,N}\frac{(2m+2\ell)!\ell!}{m!(m+\ell)!(2\ell)!}\\ &= \sum_{m=2M+1-\ell}^{\infty}(2\beta)^m\,b_{m+\ell,N}\frac{(2m+2\ell-1)!!}{(2m)!!\,(2\ell-1)!!}.\end{aligned} \qquad (5.25)$$

Using (5.19) and (5.5), we see that if $n>2M$

$$b_{n,N}=\left(\frac{c}{4}\right)^n\sum_{\ell=0}^{n}\binom{n}{\ell}a_{\ell,N}\,a_{n-\ell,N}\le\left(\frac{c}{4}\right)^n\sum_{\ell=0}^{n}\binom{n}{\ell}a_{1,N}^n\times\frac{a_{M,N}}{a_{1,N}^M}=\left(\frac{c\,a_{1,N}}{2}\right)^n\frac{a_{M,N}}{a_{1,N}^M} \qquad (5.26)$$

Therefore

$$\begin{aligned}\Delta\bar{a}_{\ell,N} &\le \frac{a_{M,N}}{a_{1,N}^M}\left(\frac{c\,a_{1,N}}{2}\right)^\ell\sum_{m=2M+1-\ell}^{\infty}(\beta c\,a_{1,N})^m\,\frac{(2m+2\ell-1)!!}{(2m)!!\,(2\ell-1)!!}\\ &\le \frac{a_{M,N}}{a_{1,N}^M}\left(\frac{c\,a_{1,N}}{2}\right)^\ell\sum_{m=2M+1-\ell}^{\infty}(\beta c\,a_{1,N})^m\binom{m+\ell}{\ell}\\ &= \frac{a_{M,N}}{a_{1,N}^M}\left(\frac{c\,a_{1,N}}{2}\right)^\ell(\beta c\,a_{1,N})^{2M+1-\ell}\sum_{k=0}^{\infty}(\beta c\,a_{1,N})^k\binom{2M+1+k}{\ell}\\ &= \frac{a_{M,N}}{a_{1,N}^M}\left(\frac{1}{2\beta}\right)^\ell(\beta c\,a_{1,N})^{2M+1}\sum_{k=0}^{\infty}(\beta c\,a_{1,N})^k\binom{2M+1+k}{\ell}.\end{aligned} \qquad (5.27)$$

Here,

$$T_{2M+1,\ell}(r) = \sum_{k=0}^{\infty} (\beta c\, a_{1,N})^k \binom{2M+1+k}{\ell} = \sum_{k=0}^{\infty} r^k \binom{2M+1+k}{\ell} = \frac{1}{1-r} \sum_{m=0}^{\ell} \binom{2M+1}{\ell-m} q^m,$$

where $r = \beta c\, a_{1,N}$, and $q = \frac{r}{1-r}$. By assumption $r < \frac{1}{2}$. The binomial coefficient in the summand is largest when $m = 0$, because $2M+1 > 2M \geq 2\ell$. Therefore,

$$T_{2M+1,\ell}(r) \leq \frac{1}{1-r}\binom{2M+1}{\ell} \sum_{m=0}^{\ell} q^m \leq \frac{1}{1-r}\frac{1}{1-q}\binom{2M+1}{\ell} = \frac{1}{1-2r}\binom{2M+1}{\ell}. \tag{5.28}$$

This proves

$$\Delta\bar{a}_{\ell,N} \leq \left(\frac{1}{2\beta}\right)^{\ell} \frac{(\beta c\, a_{1,N})^{2M+1}}{1-2\beta c\, a_{1,N}} \binom{2M+1}{\ell} \times \frac{a_{M,N}}{a_{1,N}^M} \leq \overline{\Delta\bar{a}}_{\ell,N}\,, \tag{5.29}$$

where $\overline{\Delta\bar{a}}_{\ell,N}$ is defined in (5.14). This proves $\tilde{a}_{n,N} \leq \tilde{\tilde{a}}_{n,N}$. □

Remark. We can 'improve' Proposition 5.1 by employing (correct) bounds, in a similar way as the term proportional to $\left(\frac{c\bar{a}_{1,N}}{2}\right)^n$ in (5.9). In actual calculations, we improve $\bar{a}_{n,N+1}$, $n = 1, 2, \cdots, M$, in (5.12), the upper bounds for $a_{n,N+1}$'s, using (A.6) (as well as its special case (5.5)). To be more specific, we compare $\bar{a}_{4,N+1}$ in (5.12) with $\bar{a}_{2,N+1}^2$ and replace the definition if the latter is smaller. Then we go on to 'improve' $\bar{a}_{6,N+1}$ by comparing with $\bar{a}_{2,N+1}\bar{a}_{4,N+1}$, and so on. Conceptually there is nothing really new here, but this procedure improves the actual value of the bounds in Proposition 5.1.

5.3 Computer results.

In this subsection we prove Theorem 2.2 on computers using Proposition 5.1. We double checked by Mathematica and `C++` programs on interval arithmetic. Here we will give results from `C++` programs.

Our program employs interval arithmetic, which gives rigorous bounds numerically. The idea is to express a number by a pair of 'vector', which consists of an array of length M of 'digits', taking values in $\{0, 1, 2, \cdots, 9\}$, and an integer corresponding to 'exponent'. To give a simple example, let $M = 2$. One can view that 0.0523 is expressed on the program, for example, as $I_1 = [5.2 \times 10^{-2}, 5.3 \times 10^{-2}]$, and 3 is expressed as $I_2 = [3.0 \times 10^0, 3.0 \times 10^0]$. When the division I_1/I_2 is performed, our program routines are so designed that they give correct bounds as an output. Namely, the computer output of I_1/I_2 will be $[1.7 \times 10^{-2}, 1.8 \times 10^{-2}]$. We may occasionally lose the best possible bounds, but the program is so designed that we never lose the correctness of the bounds. Thus all the outputs are rigorous bounds of the corresponding quantities.

In actual calculation we took $M = 70$ digits, which turned out to be sufficient.

We also note that interval arithmetic is employed in [14] for hierarchical model in $d = 3$ dimensions. We took independent approach in programming — we focused on ease in implementing the interval arithmetic to main programs developed for standard floating point calculations — so that structure and details of the programs are quite different. However, our numerical calculations are 'not that heavy' to require anything special.

As will be explained below, we only need to consider 2 values for the initial Ising parameter s:

$$s_- = 1.7925671170092624, \quad \text{and} \quad s_+ = 1.7925671170092625.$$

We perform explicit recursion on computers for each $s = s_\pm$ using Proposition 5.1.

We summarize what is left to be proved:

(1) $\bar{a}_{1,N} < \dfrac{1}{2\beta c}$, $0 \leq s \leq s_{N_1}$, $0 \leq N \leq N_1$, where $N_1 = 100$. This condition is from (5.15), imposed because we are going to do evaluation using Proposition 5.1. Note that this condition is stronger than (2.17) in the assumptions in Theorem 2.2, because $\dfrac{1}{2\beta c} = \dfrac{1}{2}(2+\sqrt{2}) = 1.707\cdots$ for $d = 4$.

(2) $s_- \leq \underline{s}_{N_1}$ and $\overline{s}_{N_1} \leq s_+$. To prove this, it is sufficient (as seen from the definitions (2.11) and (2.12)) to prove

$$\mu_{2,N_1} < 1, \quad \text{when} \quad s = s_- , \qquad \text{and} \quad \mu_{2,N_1} > 1 + \frac{3}{\sqrt{2}}\mu_{4,N_1} , \quad \text{when} \quad s = s_+ . \tag{5.30}$$

(3) For any s satisfying $s_- \leq s \leq s_+$, the bounds

$$(0 \leq) \quad \mu_{4,N_0} \leq 0.0045, \tag{5.31}$$

$$1.6\mu_{4,N_0}^2 \leq \mu_{6,N_0} \leq 6.07\mu_{4,N_0}^2, \tag{5.32}$$

$$(0 \leq) \quad \mu_{8,N_0} \leq 48.469\mu_{4,N_0}^3, \tag{5.33}$$

hold for $N_0 = 70$. This condition comes from the assumptions in Theorem 2.2 (sufficient, if $s_- \leq \underline{s}_{N_1}$ and $\overline{s}_{N_1} \leq s_+$).

We now summarize our results from explicit calculations.

(1) We have $\bar{a}_{1,N} \leq \frac{1}{2}s_+^2 = 1.6066\cdots$, $0 \leq s \leq s_+$, $0 \leq N \leq N_1$. The largest value for $\bar{a}_{1,N}$ in the range of parameters is actually obtained at $s = s_+$ and $N = 0$.

(2) Our calculations turned out to be accurate to obtain more than 40 digits below decimal point correctly for $\mu_{2,100}$ and $\mu_{4,100}$ at $s = s_\pm$, which is more than enough to prove (5.30). In fact, we have

$$\begin{aligned} &0.99609586499804791366176669341357334889503943 \leq \underline{a}_{1,100} \\ &\leq \mu_{2,100} \leq \bar{a}_{1,100} \leq 0.99609586499804791366176669341357334889503972 , \\ &\quad \text{at} \quad s = s_- , \end{aligned}$$

and

$$\begin{aligned} &1.01318579037206917223966110983766369438380 27 \leq \underline{a}_{1,100} \\ &\leq \mu_{2,100} \leq \bar{a}_{1,100} \leq 1.0131857903720691722396611098376636943838031 , \\ &0.0028102709780909876808879510075348013976 7915 \leq \tfrac{1}{2}(-\bar{a}_{2,100} + \underline{a}_{1,100}^{\,2}) \\ &\leq \mu_{4,100} \leq \tfrac{1}{2}(-\underline{a}_{2,100} + \bar{a}_{1,100}^2) \leq 0.0028102709780909876808879510075348013976 7969 , \\ &\quad \text{at} \quad s = s_+ . \end{aligned}$$

(3) To prove (5.31) – (5.33), we note the following. Let us write the s dependences of $a_{n,N}$ and $\mu_{n,N}$ explicitly like $a_{n,N}(s)$ and $\mu_{n,N}(s)$. For any integer N and for any s satisfying $s_- \leq s \leq s_+$, the monotonicity of $a_{n,N}(s)$ with respect to s implies

$$\mu_{4,N}(s) = \frac{1}{2}(-a_{2,N}(s) + a_{1,N}(s)^2) \leq \frac{1}{2}(-a_{2,N}(s_-) + a_{1,N}(s_+)^2) =: \bar{\mu}_{4,N} . \tag{5.34}$$

Hence if we can prove

$$\bar{\mu}_{4,70} \leq 0.0045,$$

then we have proved (5.31). In a similar way, sufficient conditions for (5.32) and (5.33) are

$$1.6 \leq \frac{\underline{\mu}_{6,70}}{\bar{\mu}_{4,70}^2} , \quad \frac{\bar{\mu}_{6,70}}{\underline{\mu}_{4,70}^2} \leq 6.07 , \quad \frac{\bar{\mu}_{8,70}}{\underline{\mu}_{4,70}^3} \leq 48.469 ,$$

with obvious definitions (as in (5.34) for $\bar{\mu}_{4,N}$) for $\underline{\mu}_{n,70}$ and $\bar{\mu}_{n,70}$.

The bounds we have for these quantities are (we shall not waste space by writing too much digits):

$$\bar{\mu}_{4,70} \leq 0.004144, \quad 3.6459 \leq \frac{\underline{\mu}_{6,70}}{\bar{\mu}_{4,70}^2} , \quad \frac{\bar{\mu}_{6,70}}{\underline{\mu}_{4,70}^2} \leq 3.7542, \quad \frac{\bar{\mu}_{8,70}}{\underline{\mu}_{4,70}^3} \leq 38.488.$$

This completes a proof of Theorem 2.2, and therefore Theorem 1.1 is proved.

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A Newman's inequalities.

Let X be a stochastic variable which is in class $\mathcal{L}$ of [15]. $X \in \mathcal{L}$ has Lee-Yang property, which states that the zeros of the moment generating function $\mathrm{E}\left[\, e^{HX} \,\right]$ are pure imaginary. In fact, it is shown in [15, Proposition 2] using Hadamard's Theorem that $\mathrm{E}\left[\, e^{HX} \,\right]$ has a following expression:

$$\mathrm{E}\left[\, e^{HX} \,\right] = e^{bH^2} \prod_j \left(1 + \frac{H^2}{\alpha_j^2}\right), \tag{A.1}$$

where b is a non-negative constant and α_j, $j = 1, 2, 3, \cdots$, is a positive nondecreasing sequence satisfying $\sum_{j=1}^{\infty} \alpha_j^{-2} < \infty$.

Consequences of (A.1) in terms of inequalities among moments (n point functions) are given in [15], among which we note the following.

1. Positivity [15, Theorem 3]. Put

$$\mu_{2n} = -\frac{1}{(2n)!} \frac{d^{2n}}{d\xi^{2n}} \log \mathrm{E}\left[\, e^{\sqrt{-1}\xi X} \,\right]\Big|_{\xi=0} . \tag{A.2}$$

Then,

$$\mu_{2n} \geq 0, \ \ n = 0, 1, 2, \cdots . \tag{A.3}$$

(Note that (A.1) implies $\mu_{2n+1} = 0$.)

2. Newman's bound [15, Theorem 6]. Put $v_{2n} = n\mu_{2n}$. Then,

$$v_{4n} \leq v_4^n, \quad v_6 \leq \sqrt{v_4 v_8}, \quad v_{4n+2} \leq v_6\, v_4^{n-1}, \tag{A.4}$$

where the first and third inequalities follow from (2.10) of [15], while the second one is (2.12) of [15]. These imply $v_{2n} \leq v_4^{n/2}$, $n \geq 2$, and therefore

$$\mu_{2n} \leq \frac{(2\mu_4)^{n/2}}{n}\,, \ \ n = 2, 3, 4, \cdots . \tag{A.5}$$

Furthermore, we will prove the following.

Proposition A.1 *Put* $a_N = \dfrac{N!}{(2N)!}\mathrm{E}\left[\, X^{2N} \,\right]$, $N \in \mathbb{Z}_+$. *Then,*

$$a_{M+N} \leq a_M\, a_N \quad N, M = 0, 1, 2, \cdots . \tag{A.6}$$

Proof. Put $y_j = \alpha_j^{-2} > 0$. Then

$$\mathrm{E}\left[\, e^{HX} \,\right] = e^{bH^2} \prod_j \left(1 + H^2\, y_j\right) . \tag{A.7}$$

Expand the infinite product to obtain

$$\prod_j \left(1 + H^2\, y_j\right) = 1 + H^2 \sum_j y_j + \frac{H^4}{2!} {\sum_{i,j}}' y_i y_j + \frac{H^6}{3!} {\sum_{i,j,k}}' y_i y_j y_k + \ldots = \sum_{n=0}^{\infty} \frac{H^{2n}}{n!} c_n, \tag{A.8}$$

with

$$c_n = \sum_{i_1, i_2, \dots, i_n}{}' y_{i_1} y_{i_2} y_{i_3} \dots y_{i_n} \ , \tag{A.9}$$

where primed summations denote summations over non-coinciding indices. Hence we have,

$$\mathrm{E}\left[\, e^{HX} \,\right] = \sum_{N=0}^{\infty} H^{2N} \sum_{m,n: m+n=N} \frac{b^m}{m!} \frac{c_n}{n!} = \sum_{N=0}^{\infty} H^{2N} \sum_{n=0}^{N} \frac{b^{N-n}}{(N-n)!} \frac{c_n}{n!} \ . \tag{A.10}$$

Comparing with $\mathrm{E}\left[\, e^{HX} \,\right] = \sum_{N=0}^{\infty} \frac{a_N}{N!} H^{2N}$, we obtain

$$a_N = \sum_{n=0}^{N} \binom{N}{n} b^{N-n} c_n \ .$$

Note that (A.9) implies

$$c_{n+m} \le c_m c_n \ , \tag{A.11}$$

because the conditions of primed summations are weaker for the left hand side. This with $b \ge 0$ implies

$$\begin{aligned}
a_M\, a_N &= \sum_{m=0}^{M} \sum_{n=0}^{N} \binom{M}{m} \binom{N}{n} b^{M+N-m-n}\, c_m\, c_n \\
&\ge \sum_{m=0}^{M} \sum_{n=0}^{N} \binom{M}{m} \binom{N}{n} b^{M+N-m-n}\, c_{m+n} \\
&= \sum_{\ell=0}^{M+N} b^{M+N-\ell}\, c_\ell \sum_{\substack{m \,:\, 0 \le m \le M, \\ 0 \le \ell - m \le N}}^{\ell} \binom{M}{m} \binom{N}{\ell - m} \\
&= \sum_{\ell=0}^{M+N} b^{M+N-\ell}\, c_\ell \binom{M+N}{\ell} = a_{M+N} \ ,
\end{aligned}$$

where, in the last line, we also used

$$\sum_{\substack{m \,:\; 0 \le m \le M, \\ 0 \le \ell - m \le N}}^{\ell} \binom{M}{m} \binom{N}{\ell - m} = \binom{M+N}{\ell} , \tag{A.12}$$

which is seen to hold if we compare the coefficients of x^ℓ of an identity $(1+x)^{M+N} = (1+x)^M (1+x)^N$. $\square$

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