Asymptotic behavior of Brownian motions on the hyperbolic spaces

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東北大学数学教室談話会

1. $\mathbf{R}^{n+1}$ 上のブラウン運動

$\{(B(t), B_{n+1}(t))\}:\ z = (x, y) \in \mathbf{R}^n \times (0, \infty)$ を出発するブラウン運動

推移確率 $P_z((B(t), B_{n+1}(t)) \in \mathrm{d}z') = (2\pi t)^{-(n+1)/2} e^{-|z'-z|^2/2t} \mathrm{d}z'.$

$0 \leqq a < y$ に対し $\tau_a = \inf\{t; B_{n+1}(t) = a\}$

$$P_z(\tau_a \in dt) = \frac{y-a}{\sqrt{2\pi t^3}} e^{-(y-a)^2/2t}\, \mathrm{d}t \qquad \text{(逆 Gauss 分布)}$$

$$P_z(B(\tau_a) \in \mathrm{d}\xi) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{y-a}{\{|\xi - x|^2 + (y-a)^2\}^{(n+1)/2}}\, \mathrm{d}\xi$$

$\xi \in \mathbf{R}^n$.　　これは Poisson 核. $n = 1$ のとき Cauchy 分布.

$u(z) = E_z[g(B(\tau_a))] = \int g(\xi) P_z(B(\tau_a) \in \mathrm{d}\xi)$ は

Poisson 方程式 $\Delta u = 0,\ u|_{y=a} = g$ の解.

Fourier 変換

$$\int_{\mathbf{R}^n} e^{\sqrt{-1}\langle u,\xi\rangle} \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{y-a}{\{|\xi - x|^2 + (y-a)^2\}^{(n+1)/2}} \mathrm{d}\xi$$

$$= e^{\sqrt{-1}\langle u,x\rangle} e^{-(y-a)|u|}, \qquad u \in \mathbf{R}^n$$

$$E_z[\exp(\sqrt{-1}\langle u, B(\tau_0)\rangle)] = e^{\sqrt{-1}\langle u,x\rangle} e^{-y|u|}$$

$$= E_z[\exp(\sqrt{-1}\langle u, B(\tau_a)\rangle) \times \exp(\sqrt{-1}\langle u, B(\tau_0) - B(\tau_a)\rangle)]$$

$$= E_{(x,y)}[\exp(\sqrt{-1}\langle u, B(\tau_a)\rangle)] \times E_{(0,a)}[\exp(\sqrt{-1}\langle u, B(\tau_0)\rangle)]$$

$$= E_{(x,y)}[\exp(\sqrt{-1}\langle u, B(\tau_a)\rangle)] \times e^{-a|u|}$$

2. Real Hyperbolic Space

$$\mathbf{H}^{n+1} = \{z = (x, y) \in \mathbf{R}^{n+1}; x \in \mathbf{R}^n, y > 0\}$$

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}, \qquad \mathrm{dvol} = y^{n+1}\mathrm{d}x\mathrm{d}y$$

$$\Delta = y^2 \sum_{j=1}^{n} \left(\frac{\partial}{\partial x^j}\right)^2 + y^2 \left(\frac{\partial}{\partial y}\right)^n - (n-1)y\frac{\partial}{\partial y}$$

heat kernels for $\Delta/2$ $\qquad p_n(t, z, z') \equiv p_t^n(r),\ r = d(z, z')$

$$p_t^2(r) = \frac{\sqrt{2}e^{-t/8}}{(2\pi t)^{3/2}} \int_r^\infty \frac{be^{-b^2/2t}}{(\cosh(b) - \cosh(r))^{1/2}}\mathrm{d}b$$

$$p_t^3(r) = \frac{1}{(2\pi t)^{3/2}} \frac{r}{\sinh(r)} \exp\left(-\frac{t}{2} - \frac{r^2}{2t}\right)$$

$$p_t^{n+2}(r) = -\frac{e^{-nt/2}}{2\pi \sinh(r)} \frac{\partial}{\partial r} p_t^n(r), \qquad \text{Millson's formula}$$

$$\Delta = y^2 \sum_{j=1}^{n} \left(\frac{\partial}{\partial x^j} \right)^2 + y^2 \left(\frac{\partial}{\partial y} \right)^n - (n-1) y \frac{\partial}{\partial y}$$

$\{Z_z(t)\}$ ：the Brownian motion on $\mathbf{H}^{n+1}$ with $Z_z(0) = z$,

i.e., the diffusion process generated by $\Delta/2$.

推移確率 $P(Z_z(t) \in \mathrm{d}z') = p(t, z, z')\ \mathrm{dvol}(z')$.

$$u(t,z) = E[g(X_z(t), Y_z(t))] = \int g(x', y') p(t, (x,y), (x', y')) \frac{\mathrm{d}x' \mathrm{d}y'}{y^{n+1}}$$

は， $\dfrac{\partial u}{\partial t} = \dfrac{1}{2} \Delta u$ **をみたす．**

解くべき確率微分方程式は， $\{(w(t), B(t))\}$ **を** Wiener **過程として**

$$\mathrm{d}X_j(t) = Y(t)\ \mathrm{d}w_j(t),$$

$$\mathrm{d}Y(t) = Y(t)\ \mathrm{d}B(t) - \frac{n-1}{2}\, Y(t)\ \mathrm{d}t, \qquad (X(0), Y(0)) = (x, y) = z.$$

$$\mathrm{d}X_j(t) = Y(t)\ \mathrm{d}w_j(t), \quad \mathrm{d}Y(t) = Y(t)\ \mathrm{d}B(t) - \frac{n-1}{2}\,Y(t)\,\mathrm{d}t.$$

解は $X_j(t) = x_j + \int_0^t Y(s)\ \mathrm{d}w_j(s),$

$$Y(t) = y\exp\left(B(t) - \frac{n}{2}t\right) \to 0,\ t \to \infty.$$

従って， $d(Z(t), Z(0)) \to \infty.$

FACT 1. Given $\{Y(t)\}$ or $\{B(t)\}$, $X_j(t)$ **の条件付確率分布は**

$$N(x_j, y^2 A_t^{(-n/2)}), \quad A_t^{(-n/2)} = \int_0^t e^{2(B(s)-ns/2)}\mathrm{d}s.$$

FACT 2. (Dufresne)

$$P\left(\frac{1}{2A_\infty^{(-n/2)}} \in \mathrm{d}u\right) = \frac{1}{\Gamma(n/2)}u^{(n/2)-1}e^{-u}\mathrm{d}u \quad \text{(Gamma **分布**)}$$

E.B.Davies (Heat Kernels and Spectral Theory)

$$\lim_{t\uparrow\infty} P\left(\frac{1-\varepsilon}{2}nt \leqq d(Z_z(t), z) \leqq \frac{1+\varepsilon}{2}nt\right) = 1, \qquad \forall\varepsilon > 0.$$

$$\cosh d(Z_z(t), z) = \frac{|X(t)-x|^2 + Y(t)^2 + y^2}{2yY(t)} = \frac{(\text{有界})}{2y}\frac{1}{y}e^{-B(t)+nt/2}$$

$$d(Z_z(t), z) \fallingdotseq \log\cosh d(Z_z(t), z) = (\text{有界}) + \frac{nt}{2} - B(t).$$

命題． $\dfrac{d(Z_z(t), z) - nt/2}{\sqrt{t}} \to N(0,1)$ as $t \to \infty$.

$X(\infty)$ **の確率分布が，** R^{n+1} **の場合の** $B(\tau_0)$ **の確率分布に対応し，**

$\mathbf{H}^{n+1}$ **上の** Poisson **核を与える．**

FACT 1. Given $\{Y(t)\}$ or $\{B(t)\}$, $X_j(t)$ の条件付確率分布は

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FACT 2. (Dufresne)

$$P\left(\frac{1}{2A_\infty^{(-n/2)}} \in \mathrm{d}u\right) = \frac{1}{\Gamma(n/2)} u^{(n/2)-1} e^{-u}\mathrm{d}u \qquad \text{(Gamma 分布)}$$

$E[\phi(X(t))]$ converges as $t \to \infty$ to

$$\int_0^\infty \frac{1}{\Gamma(n/2)} u^{(n/2)-1} e^{-u}\mathrm{d}u \int_{\mathbf{R}^n} \phi(\eta) \frac{1}{(2\pi y^2/2u)^{n/2}} e^{-|\eta-x|^2/(2y^2/2u)}\mathrm{d}\eta$$

$$= \int_{\mathbf{R}^n} \phi(\eta)\mathrm{d}\eta \int_0^\infty \frac{1}{\Gamma(n/2)} \frac{1}{\pi^{n/2} y^n} u^{n-1} e^{-(y^2+|\eta-x|^2)u/y^2}\mathrm{d}u$$

$$= \int_{\mathbf{R}^n} \phi(\eta) \ \frac{2^{n-1}\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{y^n}{(y^2+|\eta-x|^2)^n} \ \mathrm{d}\eta$$

$$\varphi_n(\eta;(x,y)) = \frac{2^{n-1}\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{y^n}{(y^2+|\eta-x|^2)^n}$$ ：Poisson 核

$n=1$ のとき $\mathbf{R}^2$ の Poisson 核と一致．

$n \geqq 2$ のときは一致しない．

Fourier 変換

$n=2$ のとき，

$$f(\lambda) = \frac{2\Gamma(3/2)}{\pi^{3/2}} \int_{\mathbf{R}^2} \frac{e^{\sqrt{-1}\langle\lambda,\eta\rangle}}{(|\eta|^2+1)^2}\,\mathrm{d}\eta = |\lambda| K_1(|\lambda|)$$ とおくと，

$$E_{(0,y)}[\exp(\sqrt{-1}\langle\lambda, X(\infty)\rangle)]$$
$$= \int_{\mathbf{R}^2} e^{\sqrt{-1}\langle\lambda,\eta\rangle}\varphi_2(\eta;(0,y))\mathrm{d}\eta = f(y\lambda), \qquad \lambda \in \mathbf{R}^2.$$

$a < y$ に対して $\tau_a = \inf\{t > 0; Y(t) = a\}$ とおくと

$$E_{(x,y)}[\exp(\sqrt{-1}\langle\lambda, X(\infty)\rangle)]$$

$$= E_{(x,y)}[\exp(\sqrt{-1}\langle\lambda, X(\tau_a)\rangle) \times \exp(\sqrt{-1}\langle\lambda, X(\infty) - X(\tau_a)\rangle)]$$

$$= E_{(x,y)}[\exp(\sqrt{-1}\langle\lambda, X(\tau_a)\rangle)] \times E_{(0,a)}[\exp(\sqrt{-1}\langle\lambda, X(\infty)\rangle)]$$

従って，$y = a$ が境界のときの Poisson 核の Fourier 変換は

$$E_{(x,y)}[\exp(\sqrt{-1}\langle\lambda, X(\tau_a)\rangle)] = e^{\sqrt{-1}\langle\lambda,x\rangle}\frac{f(y\lambda)}{f(a\lambda)}$$
$$= e^{\sqrt{-1}\langle\lambda,x\rangle}\frac{y|\lambda|K_1(y|\lambda|)}{a|\lambda|K_1(a|\lambda|)} = \int_{\mathbf{R}^2} e^{\sqrt{-1}\langle\lambda,\xi\rangle}\varphi_2((\eta, a); (x, y))\,\mathrm{d}\eta$$

Poisson 核 $\varphi_2((\eta, a); (x, y))$ の具体形？

$n = 3$ のときは $f(\lambda) = (|\lambda| + 1)e^{-|\lambda|}$

３．複素双曲空間上の Brown 運動

$\{z \in \mathbf{C}^n ; |z| < 1\}$ **上に** Bergman metric **を入れる** $(n \geqq 2)$

$$-\sum_{j,k=1}^{n} \partial_{z_j} \partial_{\overline{z_k}} \log(1-|z|^2) \mathrm{d}z_j \mathrm{d}\overline{z_k}$$

上半空間モデルを採用（Cayley **変換）**

$$\{z \in (z_1, z_2, ..., z_n) \in \mathbf{C}^n ; h(z) \equiv \mathrm{Im}(z_1) - \textstyle\sum_{k=2}^{n} |z_k|^2 > 0\}.$$

$$\mathrm{d}s^2 = -\textstyle\sum_{j,k=1}^{n} \partial_{z_j} \partial_{\overline{z_k}} \log h(z)\, \mathrm{d}z_j\, \mathrm{d}\overline{z_k}$$

Laplace-Beltrami **作用素** $\quad (y = h(z)^{1/2},\ x = \mathrm{Re}(z_1)/2)$

$$\Delta = y^2 \frac{\partial^2}{\partial y^2} + (1-2n) y \frac{\partial}{\partial y} + y^4 \frac{\partial^2}{\partial x^2} + y^2 \sum_{k=2}^{n} \left\{ \left(\frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial x} \right)^2 + \left(\frac{\partial}{\partial y_k} - x_k \frac{\partial}{\partial x} \right)^2 \right\}.$$

2点間の距離

$$\left(\cosh d(z,z')\right)^2 = \frac{(y^2+(y')^2+\sum|z_k-z'_k|^2)^2+((x'-x)+\sum(x_k y'_k - x'_k y_k))^2}{4y^2(y')^2}.$$

ブラウン運動の表示

$$Y(t) = y\exp(w_1(t)-nt)$$

$$X(t) = x + \int_0^t Y(s)^2 \mathrm{d}w_2(s) + \sum_{k=2}^{n}\int_0^t (Y_k(s)\mathrm{d}X_k(s) - X_k(s)\mathrm{d}Y_k(s))$$

$$X_k(t) = x_k + \int_0^t Y(s)\mathrm{d}w_{2k-1}(s)$$

$$Y_k(t) = y_k + \int_0^t Y(s)\mathrm{d}w_{2k}(s)$$

$t\to\infty$ **のとき，** $Y(t)\to 0$ **で残りは有限．** $d(Z(t),Z(0))\to\infty$

命題． $\dfrac{d(Z(t), Z(0)) - nt}{\sqrt{t}} \to N(0,1)$

命題． $P(X(t) \in \mathrm{d}x', X_k(t) \in \mathrm{d}x'_k, Y_k(t) \in \mathrm{d}y'_k, k = 2, ..., n)$

$$\to \frac{2^{n-1}\Gamma(n)y^n}{\pi^n(\Phi^2 + \varphi^2)^n}\ \mathrm{d}x'_2 \cdots \mathrm{d}y'_n \qquad \text{(Poisson 核)}$$

$$\Phi(y, z, z') = y^2 + \sum |z_k - z'_k|^2,$$

$$\varphi(x, x', z, z') = (x' - x) + \sum(x_k y'_k - x'_k y_k)$$

注意． $(\cosh d(z, z'))^2 = \dfrac{((y')^2 + \Phi)^2 + \varphi^2}{4y^2(y')^2}$

I. **確率面積に対する** Lévy **の公式 (確率分布の** Fourier **変換)**

II. $\left(\int_0^\infty e^{2(B_s - ns)}\mathrm{d}s,\ \int_0^\infty e^{4(B_s - ns)}\mathrm{d}s\right)$ **の同時分布の** Laplace **変換**

$0 < a < y$ に対して

$(X(\tau_a), (X_k(\tau_a), Y_k(\tau_a)))$ の確率分布の密度関数を

$m_a(\xi, \zeta), \xi \in \mathbf{R}, \zeta \in \mathbf{R}^{2(n-1)}$, とすると

$$\int_{\mathbf{R}^{2n-1}} \frac{1}{\{\Phi(a, \zeta, z')^2 + \phi(\xi, x', \zeta, z')^2\}^n} \, m_a(\xi, \zeta) \, \mathrm{d}\xi \mathrm{d}\zeta$$
$$= \frac{1}{\{\Phi(y, z, z')^2 + \phi(\xi, x', z, z')^2\}^n}$$