A limit theorem for Bohr–Jessen's probability measures of the Riemann zeta-function.

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Abstract

The asymptotic behavior of value distribution of the Riemann zeta-function $\zeta(s)$ is determined for $\frac{1}{2} < \Re(s) < 1$. Namely, the existence is proved, and the value is given, of the limit

$$\lim_{\ell\to\infty} (\ell\,(\log \ell)^{\sigma})^{-1/(1-\sigma)} \log W(\mathbf{C}\setminus R(\ell), \sigma, \zeta)$$

for $\frac{1}{2} < \sigma < 1$, where $R(\ell)$ is a square in the complex plane $\mathbf{C}$ of side length 2ℓ centered at 0, and

$$W(A,\sigma,\zeta) = \lim_{T\to\infty} (2T)^{-1}\mu_1(\{t \in [-T,T] \mid \log\zeta(\sigma + t\sqrt{-1}) \in A\})\,, \quad A \subset \mathbf{C}\,,$$

where μ_1 is the one-dimensional Lebesgue measure. Analogous results are obtained also for the Dedekind zeta-functions of Galois number fields.

As an essential step, a limit theorem for a sum of independent random variables $X = \sum_{n=1}^{\infty} r_n X_n$ is proved, where X_n, $n \in \mathbf{N}$, have identical distribution on a finite interval with mean zero, and $\{r_n\}$ is a regularly varying sequence of index $-\sigma$. The limit theorem states the convergence of

$$\lim_{N\to\infty} N^{-1} \log \mathrm{Prob}[\, X > \sum_{n\le N} r_n \,]$$

and gives the explicit value of the limit. In particular, it is shown that the value depends only on σ and is otherwise independent of $\{r_n\}$.

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1 Introduction.

Let $s = \sigma + i\,t \in \mathbf{C}$ ($i = \sqrt{-1}$) be a complex variable, and $\zeta(s)$ the Riemann zeta-function. Let μ_1 be the one-dimensional Lebesgue measure. Bohr-Jessen [1] proved the existence of the limit

$$W(A,\sigma,\zeta) = \lim_{T\to\infty}(2T)^{-1}\mu_1(\{t \in [-T,T] \mid \log\zeta(\sigma + t\sqrt{-1}) \in A\})\,,$$

for $\sigma > 1/2$ and for closed rectangle $A \subset \mathbf{C}$ with the edges parallel to the axes. We can extend $W(A,\sigma,\zeta)$ to a probability measure defined on $\mathbf{C}$.

Consider the particular case where A is the complement of a square centered at the origin;

$$A = \mathbf{C}\setminus R(\ell)\,;\quad R(\ell) = \{z \in \mathbf{C} \mid \max\{|\Re(z)|,\ |\Im(z)|\} \le \ell\}\,,\ \ell > 0\,.$$

If $\sigma > 1$, $W(\mathbf{C}\setminus R(\ell),\sigma,\zeta) = 0$ for sufficiently large ℓ, because the Euler product expansion of $\zeta(s)$ is uniformly convergent. It is therefore the case $\frac{1}{2} < \sigma \le 1$ for which the behavior of $W(\mathbf{C}\setminus R(\ell),\sigma,\zeta)$ raises an interesting problem.

The main purpose of this paper is to prove the following.

Theorem 1. *For $\frac{1}{2} < \sigma < 1$ it holds that*

$$W(\mathbf{C}\setminus R(\ell),\sigma,\zeta) = \exp\left(-A(\sigma)\{\ell\,(\log\ell)^{\sigma}\}^{\frac{1}{1-\sigma}}(1+o(1))\right)$$

as ℓ tends to infinity, where

$$A(\sigma) = (1-\sigma)\,\{\frac{1-\sigma}{\sigma}\int_0^\infty \log I_0(y^{-\sigma})\,dy\}^{-\frac{\sigma}{1-\sigma}}\,,$$

and $I_0(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\exp(x\cos u)\,du$ is the zeroth modified Bessel function.

This theorem gives the precise form of the exponent in the asymptotic behavior of the value distribution $1 - W(R(\ell),\sigma,\zeta) = W(\mathbf{C}\setminus R(\ell),\sigma,\zeta)$ of Riemann zeta-function $\zeta(s)$ for $\frac{1}{2} < \sigma < 1$. Professor E. Bombieri pointed out that $A(\sigma) \sim K\,(1-\sigma)$ (with an absolute constant K) as $\sigma \to 1-$, and $A(\sigma) \sim 2\,(2\sigma - 1)$ as $\sigma \to \frac{1}{2}+$. We show this fact in the Appendix.

The progress in the quantitative study of $W(R(\ell),\sigma,\zeta)$ was slow. For a long time, the only non-trivial result was the upper bound

$$W(\mathbf{C}\setminus R(\ell),\sigma,\zeta) = O(\exp(-\lambda\ell^2))$$

due to Jessen–Wintner [4], where λ is any positive constant. Nikishin considered a closely related problem [15], whose result suggested an improvement in the exponent of ℓ in the right-hand side of the above equation (see Laurinčikas [10, Chapter 3]).

The best known result has been a bound of the form

$$\begin{aligned}&\exp\left(-M_1(\sigma,\zeta)\{\ell\,(\log\ell)^{\sigma}\}^{\frac{1}{1-\sigma}}(1+o(1))\right) \le W(\mathbf{C}\setminus R(\ell),\sigma,\zeta)\\ &\le \exp\left(-M_2(\sigma,\zeta)\{\ell\,(\log\ell)^{\sigma}\}^{\frac{1}{1-\sigma}}(1+o(1))\right),\end{aligned}$$

where $M_1(\sigma,\zeta)$ and $M_2(\sigma,\zeta)$ are positive constants. This kind of bounds was first proved by Joyner [5, Chapter 5, Theorem 4.3], and the values of $M_1(\sigma,\zeta)$ and $M_2(\sigma,\zeta)$ were improved in subsequent works of Matsumoto [12] and Hattori–Matsumoto [3]. Those *inequalities* can now be replaced by the *equality* of the above Theorem 1.

It is possible to generalize this theorem to Dedekind zeta-functions of Galois number fields, which will be discussed in Section 4.

A main new ingredient in the proof of Theorem 1 is an asymptotic estimate, developed in Section 2, of a sum of independent random variables $X = \sum_{n=1}^{\infty} r_n X_n$, where X_n, $n \in \mathbf{N}$, have identical distribution on a finite interval with mean zero, and $\{r_n\}$ is a regularly varying sequence of index $-\sigma$ [16]. The series is convergent (but not absolutely convergent) if $\frac{1}{2} < \sigma < 1$, for which case the limit $\lim_{N\to\infty} N^{-1} \log \mathrm{Prob}[\ X > \sum_{n\le N} r_n\ ]$ is shown to exist, and its value given in Theorem 2. In particular, it is shown that the value depends only on σ and is otherwise independent of $\{r_n\}$. A Tauberian theorem of exponential type [9] is applied, which relates the above quantity to the Laplace transform $\mathrm{E}[\ \exp(yX)\ ]$. A Riemann-like integration is introduced to calculate the Laplace transform (Lemma 4), where the regularly varying property of $\{r_n\}$ is essential. Except for this Lemma, the regularly varying property of $B_N = \sum_{n\le N} r_n$ is sufficient. The results of Section 2 are independent of number theoretic arguments, and can be read independently of the rest of the paper.

In Section 3 we apply Theorem 2 to the case $r_n = p_n^{-\sigma}$, where p_n is the n-th prime number. The result is known [5] [12] to be related to $W(\mathbf{C}\setminus R(\ell),\sigma,\zeta)$, whence Theorem 1 is proved. The prime number theorem plays an essential role in Section 3, most crucially in the fact that $\{p_n\}$ is a regularly varying sequence, implying an applicability of Theorem 2.

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2 Limit theorem for a sum of independent random variables.

A positive and measurable function $\phi(x)$ defined for sufficiently large positive x is called a regularly varying function with index α ([6] [7] [8] [16]), if

$$(1)\quad \lim_{x\to\infty} \frac{\phi(\lambda x)}{\phi(x)} = \lambda^{\alpha},$$

for any $\lambda > 1$. It is known ([2] [16]) that a regularly varying function $\phi(x)$ has an (asymptotically uniquely determined) asymptotic inverse $\psi(x)$, that is, a function satisfying

$$\lim_{x\to\infty} \frac{1}{x}\, \psi(\phi(x)) = 1\,.$$

In this section, we prove the following limit theorem for a sum of independent random variables.

Theorem 2. *Let* $\{X_n\}$*,* $n \in \mathbf{N}$*, be independent random variables with identical distribution satisfying* $\mathrm{E}[\ X_1\ ] = 0$*,* $\mathrm{V}[\ X_1\ ] > 0$*, (where* $\mathrm{E}[\ \cdot\ ]$ *denotes the expectation value, and* $\mathrm{V}[\ \cdot\ ]$ *the variance) and* $|X_1| \le M$ *almost surely for some* $M > 0$*. Let* σ *be a constant satisfying* $\frac{1}{2} < \sigma < 1$*, and let* $\{r_n\}$*,* $n \in \mathbf{N}$*, be a sequence of positive numbers, such that* $r_{[x]}$ *(where* $[x]$ *denotes the largest integer not exceeding* x*) is a regularly varying function with index* $-\sigma$*. Then* $X = \sum_{n\ge1} r_n X_n$ *converges almost surely, and*

$$\text{(2)}\qquad \lim_{N\to\infty} N^{-1} \log \mathrm{Prob}[\ X > \sum_{n\le N} r_n\ ] = -G(\sigma),$$

where

$$\text{(3)}\qquad G(\sigma) = \left\{ \frac{1-\sigma}{\sigma} \int_0^\infty \log I(y^{-\sigma})\, dy \right\}^{-\sigma/(1-\sigma)},$$

with $I(x) = \mathrm{E}[\ \exp(x\,X_1)\ ]$*,* $x \in \mathbf{R}$*.*

Remark. The integral in (3) is convergent, because $\log I(y^{-\sigma}) = O(y^{-\sigma})$ as $y \to +0$ and $\log I(y^{-\sigma}) = O(y^{-2\sigma})$ as $y \to +\infty$.

The main tool for the proof of Theorem 2 is the following Tauberian theorem of exponential type, which was proved by Kasahara [9].

Lemma 3 [9, Cor. 1(i) to Theorem 2] . *Let* X *be a real random variable satisfying* $\mathrm{Prob}[\ X > a\ ] > 0$ *for any* $a > 0$*,* $\phi(x)$ *be a regularly varying function with index* $1-\sigma$*,* $\psi(x)$ *be an asymptotic inverse of* $\dfrac{x}{\phi(x)}$*. Then*

$$\lim_{x\to\infty} \frac{1}{x} \log \mathrm{Prob}[\ X > \phi(x)\ ] = -G < 0$$

holds if and only if

$$\lim_{\lambda\to\infty} \frac{1}{\psi(\lambda)} \log \mathrm{E}[\ \exp(\lambda X)\ ] = \sigma \left(\frac{1-\sigma}{G}\right)^{(1-\sigma)/\sigma}.$$

Remark. The original result in [9] is written in terms of $\mu(\cdot) = \mathrm{Prob}[\ X \in \cdot\ ]$. There it is assumed that μ is a finite measure on $(0,\infty)$. In terms of X, this implies $X > 0$. However, it is easy to see in [9] that this assumption can be removed. Our notation is consistent with that of [9] if we put $\sigma = 1-\alpha$ and $G = A$ in Lemma 3.

Proof of Theorem 2. First we show the almost sure convergence of X. The regularly varying property of $\{r_n\}$ implies that for any $\epsilon > 0$ there exists a constant $C > 0$ such that $r_n \le C\, n^{-\sigma+\epsilon}$, $n \in \mathbf{N}$ [16, §1.5]. Hence it follows that $\sum_{n\ge 1} r_n^2 < \infty$ for $\sigma > \frac{1}{2}$, which, with $\mathrm{E}[\, X_1 \,] = 0$ and the Kolmogorov's theorem, implies that X converges almost surely. (Note that $\sigma < 1$ implies $\sum_{n\ge 1} r_n = \infty$, which further implies that X is not absolutely convergent.)

Let $B_x = \sum_{n\le x} r_n$. Applying [8] [16, Theorem 2.1, Exercise 2.1], we have

$$B_x = \int_0^x r_{[u]}\, du = \frac{1}{1-\sigma}\, x\, r_{[x]}\,(1+o(1))\,, \tag{4}$$

hence B_x is a regularly varying function with index $1-\sigma$. Therefore we can apply Lemma 3 with $\phi(x) = B_x$. Note that the definitions of X and $I(x)$ imply

$$\log \mathrm{E}[\; \exp(x\, X) \;] = \sum_{n\ge 1} \log I(x\, r_n)\,.$$

Using this relation and (3), we see from Lemma 3 that it is sufficient to prove

$$\lim_{\lambda\to\infty} (1-\sigma)^{-1/\sigma}\, \psi(\lambda)^{-1} \sum_{n\ge 1} \log I(\lambda\, r_n) = \int_0^\infty \log I(y^{-\sigma})\, dy\,, \tag{5}$$

where $\psi(x)$ is an asymptotic inverse of $\dfrac{x}{B_x}$. Using (4) we have

$$1 = \lim_{x\to\infty} x^{-1}\, \psi\left(\frac{x}{B_x}\right) = \lim_{x\to\infty} x^{-1}\, \psi((1-\sigma)\, {r_{[x]}}^{-1})\,.$$

Note that $\psi(x)$ is a regularly varying function with index $1/\sigma$. It is known [16, Theorem 1.1] that the convergence in the defining equation (1) is in fact a uniform convergence for λ in any finite closed interval of $(0,\infty)$. Hence,

$$(1-\sigma)^{1/\sigma} \lim_{x\to\infty} x^{-1}\, \psi({r_{[x]}}^{-1}) = \lim_{x\to\infty} x^{-1}\, \psi((1-\sigma)\, {r_{[x]}}^{-1}) = 1\,. \tag{6}$$

This implies that the function $(1-\sigma)^{1/\sigma}\, \psi(x^\sigma)$ is an asymptotic inverse of ${r_{[x]}}^{-1/\sigma}$.

It is now easy to see that (5) is a consequence of the following Lemma 4, with $f(y) = \log I(y^{-\sigma})$, $q(x) = {r_{[x]}}^{-1/\sigma}$, and $\rho(x) = (1-\sigma)^{1/\sigma}\, \psi(x^\sigma)$. □

Lemma 4. *Let $f(y)$ be a continuous, non-negative, decreasing function on $y > 0$, satisfying $f(y) = O(y^{-\sigma})$, $y \to 0$, and $f(y) = O(y^{-2\sigma})$, $y \to \infty$, where σ is a constant satisfying $\frac{1}{2} < \sigma < 1$. Let $q(x)$ be a positive function defined for $x \ge 1$, regularly varying with index 1, and $\rho(x)$ be one of its asymptotic inverse. Then*

$$\lim_{x\to\infty} \frac{1}{\rho(x)} \sum_{n\ge 1} f\left(\frac{q(n)}{x}\right) = \int_0^\infty f(y)\, dy\,.$$

Remark. The assumptions imply that f is Riemann integrable at $(0, \infty)$.

Proof of Lemma 4. The function $q(x)$ is a regularly varying function with index 1, hence has a representation [16, Theorem 1.2] for sufficiently large x, say $x \geq x_0$, of a form

$$(7) \quad q(x) = x \exp\{\eta(x) + \int_{x_0}^{x} \frac{\epsilon(u)}{u}\, du\}\,, \quad x \geq x_0,$$

where $\eta(x)$ is a function which converges as $x \to \infty$, and $\epsilon(x)$ is a continuous function which converges to 0 as $x \to \infty$. Put $c = \lim_{x\to\infty} \eta(x)$, and define a continuous function $Q(x)$ on $x \geq x_0$ by

$$(8) \quad Q(x) = x \exp\{c + \int_{x_0}^{x} \frac{\epsilon(u)}{u}\, du\}\,.$$

In particular, $Q(x)$ is strictly increasing for sufficiently large x, and $\lim_{x\to\infty} Q(x) = \infty$. Using the uniform convergence of (1) as in the argument for (6) in the proof of Theorem 2, we have

$$\lim_{x\to\infty} x^{-1} \rho(Q(x)) = 1\,.$$

Therefore it is sufficient to prove

$$(9) \quad \lim_{x\to\infty} \frac{1}{x} \sum_{n\geq 1} f\left(\frac{q(n)}{Q(x)}\right) = \int_0^\infty f(y)\, dy\,.$$

The assumptions on f imply

$$\int_1^\infty f(y^{1-\epsilon_0})\, dy < \infty\,, \quad \text{if } 0 < \epsilon_0 < 1 - (2\sigma)^{-1}.$$

Fix such an ϵ_0. The properties of $\eta(x)$ and $\epsilon(x)$ imply that there exists a constant $x_1 \geq x_0$ such that if $x \geq x_1$ then $\exp(\eta(x) - c) \geq 2^{-1}$ and $|\epsilon(x)| \leq \epsilon_0$. Assume $b \geq 1$, $x \geq x_1$ and $n > b\,x$ $(\geq x_1)$. It then follows from (7) and (8) that

$$\frac{q(n)}{Q(x)} \geq \frac{1}{2}\left(\frac{n}{x}\right)^{1-\epsilon_0}\,.$$

Monotonicity and non-negativity of f therefore imply that

$$0 \leq \frac{1}{x} \sum_{n>b\,x} f\left(\frac{q(n)}{Q(x)}\right) \leq \frac{1}{x} \sum_{n>b\,x} f\left(\frac{1}{2}\left(\frac{n}{x}\right)^{1-\epsilon_0}\right) \leq \int_{b-1}^\infty f\left(\frac{1}{2}y^{1-\epsilon_0}\right) dy \quad (< \infty)\,.$$

The right hand side of this inequality converges to 0 as $b \to \infty$ uniformly in x. Hence,

$$(10) \quad \lim_{b\to\infty} \frac{1}{x} \sum_{n>b\,x} f\left(\frac{q(n)}{Q(x)}\right) = 0\,, \quad \text{uniformly in } x \geq x_1.$$

By assumption, there exists a positive constant C such that

$$(11)\quad f(y) \leq C\,\frac{1}{y^{\sigma}}\,,\quad 0 < y \leq 1.$$

Let ϵ_1 be a constant satisfying

$$(12)\quad 0 < \epsilon_1 < \sigma^{-1} - 1\,.$$

The assumptions on f imply $\int_0^1 f(y^{1+\epsilon_1})\,dy < \infty$. In a similar way as the arguments following (9), it follows from (7), (8), and the properties of $\eta(x)$ and $\epsilon(x)$ that there exist constants x_2 and $C' > 0$ such that

$$(13)\quad \frac{q(n)}{Q(x)} \geq \frac{1}{2}\left(\frac{n}{x}\right)^{1+\epsilon_1}\,,\quad x_2 \leq n \leq x,$$

and

$$(14)\quad Q(x) \leq C'\,x^{1+\epsilon_1}\,,\quad x \geq x_2\,.$$

Put $C_q = \min_{1\leq n\leq x_2} q(n) > 0$. As noted above, $Q(x)$ is increasing for sufficiently large x and $\lim_{x\to\infty} Q(x) = \infty$. Therefore, there exists a constant $x_3 \geq x_2$ such that

$$Q(y) \geq C_q \text{ and } \sup_{0<x\leq y} Q(x) \leq Q(y)\,,\quad y \geq x_3\,.$$

With monotonicity of f and (11), it then follows that if $1 \leq n \leq x_2$ and $0 < x \leq y$ for some $y \geq x_3$, then

$$f\left(\frac{q(n)}{Q(x)}\right) \leq f\left(\frac{C_q}{Q(y)}\right) \leq C\left(\frac{Q(y)}{C_q}\right)^{\sigma}\,.$$

With (14), we see that there exists a positive constant C'' such that if $y \geq x_3$,

$$(15)\quad f\left(\frac{q(n)}{Q(x)}\right) \leq C''\,y^{(1+\epsilon_1)\,\sigma}\,,\quad 1 \leq n \leq x_2,\ 0 < x \leq y.$$

Let a be a small positive number. We now prove

$$(16)\quad \lim_{a\to 0}\frac{1}{x}\sum_{1\leq n<a\,x} f\left(\frac{q(n)}{Q(x)}\right) = 0\,,\quad \text{uniformly in } x \geq x_3.$$

We consider two cases $x_3 \leq x < \frac{x_2}{a}$ and $x \geq \max\{\frac{x_2}{a}, x_3\}$ separately.
(i) The case $x_3 \leq x < \frac{x_2}{a}$. This occurs only when $\frac{x_2}{a} > x_3$. Then, using (15) with $y = x_2/a$, we see that

$$(17)\quad \frac{1}{x}\sum_{1\leq n<ax} f\left(\frac{q(n)}{Q(x)}\right) \leq C''\,{x_2}^{(1+\epsilon_1)\,\sigma}\,a^{1-(1+\epsilon_1)\,\sigma}\,.$$

(ii) The case $x \geq \max\{\frac{x_2}{a}, x_3\}$. In this case we use (15) with $y = x$ to see that

$$(18)\quad \frac{1}{x}\sum_{1\leq n<x_2} f\left(\frac{q(n)}{Q(x)}\right) \leq x_2\,C''\,x^{(1+\epsilon_1)\,\sigma-1} \leq C''\,{x_2}^{(1+\epsilon_1)\,\sigma}\,a^{1-(1+\epsilon_1)\,\sigma}\,,$$

where we used (12), which implies $(1+\epsilon_1)\,\sigma - 1 < 0$, in the last inequality. On the other hand, monotonicity of f implies, with (13),

$$(19)\quad \frac{1}{x}\sum_{x_2 \le n < a\,x} f\left(\frac{q(n)}{Q(x)}\right) \le \int_0^a f\left(\frac{1}{2}\,y^{1+\epsilon_1}\right)\,dy \quad (<\infty).$$

Combining (18) and (19), we have,

$$(20)\quad \frac{1}{x}\sum_{1 \le n < ax} f\left(\frac{q(n)}{Q(x)}\right) \le C''\,{x_2}^{(1+\epsilon_1)\,\sigma}\,a^{1-(1+\epsilon_1)\,\sigma} + \int_0^a f\left(\frac{1}{2}\,y^{1+\epsilon_1}\right)\,dy\,,$$

for $x \ge \max\{\frac{x_2}{a}, x_3\}$.

The results of the two cases (17) and (20), together with non-negativity of f, imply our claim (16).

The three equations (9), (16), and (10) imply that it is sufficient to prove

$$(21)\quad \lim_{x\to\infty} \frac{1}{x}\sum_{n;\; a\,x \le n \le b\,x} f\left(\frac{q(n)}{Q(x)}\right) = \int_a^b f(y)\,dy\,, \quad 0 < a < b < \infty\,.$$

Fix $a > 0$ and $b > a$. The properties of $\eta(x)$ and $\epsilon(x)$ in (7) imply that $|\eta(n) - c + \int_x^n \frac{\epsilon(u)}{u}\,du|$ can be made arbitrarily small as x is increased, if $a\,x \le n \le b\,x$. Therefore, (8) and (7) imply that for any $\delta > 0$ there exists $x_4 = x_4(\delta)$ such that

$$(22)\quad \left|\frac{q(n)}{Q(x)} - \frac{n}{x}\right| \le \delta\,, \quad x \ge x_4\,,\ a\,x \le n \le b\,x\,.$$

On the other hand, the continuity of f implies that it is in fact uniformly continuous on a closed interval $[a/2, 2b]$. Namely, for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$(23)\quad |f(y) - f(y')| \le \epsilon\,, \quad |y - y'| \le \delta\,,\ \frac{a}{2} \le y,\, y' \le 2b.$$

With (22) and (23) we have

$$\frac{1}{x}\sum_{a\,x \le n \le b\,x} \left| f\left(\frac{q(n)}{Q(x)}\right) - f\left(\frac{n}{x}\right)\right| \le \epsilon\,(b - a + 1)\,,$$

for sufficiently large x. Therefore,

$$\begin{aligned} &\left|\frac{1}{x}\sum_{n;\; a\,x \le n \le b\,x} f\left(\frac{q(n)}{Q(x)}\right) - \int_a^b f(y)\,dy\right| \\ &\le \frac{1}{x}\sum_{n;\; a\,x \le n \le b\,x} \left| f\left(\frac{q(n)}{Q(x)}\right) - f\left(\frac{n}{x}\right)\right| + \left|\frac{1}{x}\sum_{n;\; a\,x \le n \le b\,x} f\left(\frac{n}{x}\right) - \int_a^b f(y)\,dy\right| \\ &\to 0, \quad x \to \infty, \end{aligned}$$

where we also used the integrability of f. This proves (21), hence the Lemma. □

3 Proof of Theorem 1.

For $n = 1, 2, 3, \cdots$, let p_n be the n-th prime number. The prime number theorem

$$\lim_{x\to\infty} \frac{\log x}{x} \sum_{n;\ p_n \le x} 1 = 1\,,$$

implies $\lim_{n\to\infty} n\, {p_n}^{-1} \log p_n = 1$, from which we see

$$(24) \quad \lim_{n\to\infty} p_n\, (n \log n)^{-1} = 1\,.$$

In particular, this implies that $p_{[x]}$ is a regularly varying function with index 1.

Let θ_n, $n \in \mathbf{N}$, be i.i.d. random variables each with uniform distribution on $[0, 1)$, and put

$$(25) \quad X = \sum_{n\ge 1} {p_n}^{-\sigma} \cos(2\pi\theta_n)\,,$$

$$(26) \quad Y = \sum_{n\ge 1} {p_n}^{-\sigma} \sin(2\pi\theta_n)\,.$$

Theorem 2 with $X_n = \cos(2\pi\theta_n)$ and $r_n = {p_n}^{-\sigma}$ implies, for $\frac{1}{2} < \sigma < 1$,

$$(27) \quad \lim_{N\to\infty} \frac{1}{N} \log \mathrm{Prob}[\, X > B_N \,] = -\left\{ \frac{1-\sigma}{\sigma}\, J(\sigma) \right\}^{\frac{-\sigma}{1-\sigma}}, \quad J(\sigma) = \int_0^\infty \log I_0(y^{-\sigma})\, dy,$$

where $B_N = \sum_{n\le N} {p_n}^{-\sigma}$ and $I_0(x)$ is the zeroth modified Bessel function.

We here recall a fact [12, §4] (notations are changed; see the remark below) that there exists a constant A (independent of ℓ) such that,

$$(28) \quad W(\mathbf{C} \setminus R(\ell), \sigma, \zeta) \le \mathrm{Prob}[\, |X| > \ell - A \,] + \mathrm{Prob}[\, |Y| > \ell - A \,]\,,$$

and

$$(29) \quad W(\mathbf{C} \setminus R(\ell), \sigma, \zeta) \ge \mathrm{Prob}[\, |X| > \ell + A \,]\,,$$

for $\ell > A$.

Define $N = N(\ell)$ by

$$(30) \quad B_N < \ell - A \le B_{N+1}\,.$$

As noted in the proof of Theorem 2 (see (4)), B_N is a regularly varying function with index $1-\sigma > 0$, which implies $\lim_{N\to\infty} B_N = \infty$, hence $N(\ell)$ is well-defined for sufficiently large ℓ. Furthermore, (4) (with $r_n = {p_n}^{-\sigma}$) and (24) imply $\lim_{x\to\infty} B_x\, (x \log x)^{\sigma} x^{-1} = (1-\sigma)^{-1}$, which, with (30), implies $\lim_{\ell\to\infty} \ell\, N(\ell)^{\sigma-1} (\log N(\ell))^{\sigma} = (1-\sigma)^{-1}$, hence we have,

$$N(\ell) = (1-\sigma)\, \ell^{\frac{1}{1-\sigma}}\, (\log \ell)^{\frac{\sigma}{1-\sigma}}\, (1 + o(1))\,.$$

This and (27) imply

$$(31)\quad \lim_{\ell\to\infty} \ell^{\frac{-1}{1-\sigma}} (\log \ell)^{\frac{-\sigma}{1-\sigma}} \log \mathrm{Prob}[\, |X| > B_N \,] = -(1-\sigma) \left\{ \frac{1-\sigma}{\sigma} J(\sigma) \right\}^{-\sigma/(1-\sigma)} .$$

where we also used $\mathrm{Prob}[\, X < -B_N \,] = \mathrm{Prob}[\, X > B_N \,]$. Note also that

$$Y = \sum_{n\geq 1} p_n{}^{-\sigma} \sin(2\pi\theta_n) = \sum_{n\geq 1} p_n{}^{-\sigma} \cos(2\pi(\theta_n - \frac{1}{4}))$$

implies

$$(32)\quad \mathrm{Prob}[\, |Y| > B_N \,] = \mathrm{Prob}[\, |X| > B_N \,] .$$

Substituting (31) and (32) in (28), we obtain

$$(33)\quad \limsup_{\ell\to\infty} \ell^{\frac{-1}{1-\sigma}} (\log \ell)^{\frac{-\sigma}{1-\sigma}} \log W(\mathbf{C} \setminus R(\ell), \sigma, \zeta) \leq -(1-\sigma) \left\{ \frac{1-\sigma}{\sigma} J(\sigma) \right\}^{-\sigma/(1-\sigma)} .$$

Similarly, if we define $N' = N'(\ell)$ by $B_{N'-1} \leq \ell + A < B_{N'}$, we have, with (29),

$$(34)\quad \liminf_{\ell\to\infty} \ell^{\frac{-1}{1-\sigma}} (\log \ell)^{\frac{-\sigma}{1-\sigma}} \log W(\mathbf{C} \setminus R(\ell), \sigma, \zeta) \geq -(1-\sigma) \left\{ \frac{1-\sigma}{\sigma} J(\sigma) \right\}^{-\sigma/(1-\sigma)} .$$

The two bounds (33) and (34) prove Theorem 1. □

Remark. Our notations B_N, ℓ, A are written as $\sum_{n=1}^{N} |B_n|$, r, $A + \eta$ in [12]. The value of η is specified in [12], but the results quoted here are valid for any $\eta > 0$.

4 Dedekind zeta-functions of Galois number fields.

Let F be a Galois extension of the rational number field $\mathbf{Q}$, $d = [F : \mathbf{Q}] \geq 2$, and $\zeta_F(s)$ the corresponding Dedekind zeta-function. The existence of the limit

$$W(\mathbf{C} \setminus R(\ell), \sigma, \zeta_F) = \lim_{T\to\infty} (2T)^{-1} \mu_1(\{ t \in [-T, T] \mid \log \zeta_F(\sigma + t\sqrt{-1}) \in \mathbf{C} \setminus R(\ell) \})$$

for $\sigma > 1 - d^{-1}$ was proved in Matsumoto [11] or [13], and Joyner's type inequality for $W(\mathbf{C} \setminus R(\ell), \sigma, \zeta_F)$ was given in Matsumoto [12]. In this section, as a generalization of Theorem 1, we prove the following theorem, which implies that the asymptotic behavior of $W(\mathbf{C} \setminus R(\ell), \sigma, \zeta_F)$ is ruled only by σ and the degree d.

Theorem 5. *Let F be a Galois extension of $\mathbf{Q}$, and $d = [F : \mathbf{Q}] \geq 2$. Then, for $1 - d^{-1} < \sigma < 1$ it holds that*

$$\lim_{\ell\to\infty} \{ \ell (\log \ell)^{\sigma} \}^{-\frac{1}{1-\sigma}} \log W(\mathbf{C} \setminus R(\ell), \sigma, \zeta_F) = -\frac{1}{d} A(\sigma) ,$$

where $A(\sigma)$ is given in Theorem 1.

Proof. Let p_k be the k-th prime number, and $\wp_k^{(j)}$ $(1 \le j \le g(k))$ be all prime divisors of p_k in F. Define the integers $e(k)$ and $f(k)$ by

$$p_k = \prod_{j=1}^{g(k)} (\wp_k^{(j)})^{e(k)}, \quad N\wp_k^{(j)} = p_k^{f(k)},$$

where $N\wp_k^{(j)}$ is the norm of $\wp_k^{(j)}$. Let

$$\beta_k = \begin{cases} g(k)\, p_k{}^{-\sigma} & \text{if } f(k) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In the case of $\zeta_F(s)$, instead of (25) and (26), we should consider

$$\begin{aligned} X_F &= \sum_{k\ge1} \beta_k \cos(2\pi\theta_k) = \sum_{n\ge1} \beta_{m(n)} \cos(2\pi\theta_{m(n)})\,, \\ Y_F &= \sum_{k\ge1} \beta_k \sin(2\pi\theta_k) = \sum_{n\ge1} \beta_{m(n)} \sin(2\pi\theta_{m(n)}) \end{aligned}$$

(see (4.1) and (5.1) in [12]), where $m(1)$, $m(2)$, $\cdots$ is the monotonically increasing sequence of all k's such that $f(k) = 1$.

We claim that $\beta_{m([x])}$ is a regularly varying function with index $-\sigma$. In fact, since there exist only finitely many k's such that $e(k) > 1$, we can find an N_0 for which $e(m(n)) = 1$, hence $g(m(n)) = d$, holds for any $n > N_0$. For such an n we see that

$$n = \sum_{\substack{p_k \le p_{m(n)} \\ g(k) = d}} 1 + O(1) = \frac{1}{d}\,\frac{p_{m(n)}}{\log p_{m(n)}}\,(1 + o(1))$$

by virtue of the prime ideal theorem. Hence it follows that

$$p_{m(n)} = d\,n\,\log n\,(1 + o(1)),$$

from which our claim follows easily.

Therefore we can apply Theorem 2 with $X_n = \cos(2\pi\theta_{m(n)})$ and $r_n = \beta_{m(n)}$ to obtain

$$\lim_{N\to\infty} \frac{1}{N} \log \mathrm{Prob}[\, X_F > B_{N,F}\,] = -\left(\frac{1-\sigma}{\sigma}\, J(\sigma)\right)^{\frac{-\sigma}{1-\sigma}},$$

where $B_{N,F} = \sum_{n\le N} \beta_{m(n)}$.

The rest of the proof proceeds along the lines of that of Theorem 1. The inequalities (28) and (29) are valid with the replacements, ζ, X, Y, by ζ_F, X_F, Y_F, respectively. The integer $N = N(\ell, F)$ is defined by $B_{N,F} < \ell - A \le B_{N+1,F}$. Then, since

$$\begin{aligned} B_{N,F} &= \frac{1}{1-\sigma} m(N)^{1-\sigma} (\log m(N))^{-\sigma} (1 + o(1)), \\ m(N) &= dN\,(1 + o(1)), \\ B_{N+1,F} &= B_{N,F}\,(1 + o(1)), \end{aligned}$$

we obtain

$$N(\ell, F) = \frac{1-\sigma}{d}\,\ell^{\frac{1}{1-\sigma}}(\log \ell)^{\frac{\sigma}{1-\sigma}}(1+o(1)),$$

from which it follows that the upper-bound corresponding to (33), with ζ replaced by ζ_F and the right-hand side divided by d. The lower-bound part is similar (the rearrangement argument in [12] causes no problem here), hence the proof of Theorem 5 is completed. □

Joyner's type inequalities are known for some other cases such as Dedekind zeta-functions of non-Galois number fields [12] and zeta-functions attached to certain cusp forms ([3] [12] [14]). Moreover, similar inequalities can be shown on the line $\sigma = 1$ ([12]). It is an interesting problem to generalize our asymptotic results to these cases. However, since our proofs in the present paper heavily depend on the properties of regularly varying functions with positive exponent, some new idea seems to be required to challenge this problem.

Appendix

Here we shall give Bombieri's proof of the asymptotic behavior of $A(\sigma)$ mentioned directly after the statement of Theorem 1.

Put $F(x) = \dfrac{I_0'(x)}{I_0(x)}$. After a change of variables $x = y^{-\sigma}$ and an integration by parts, we have

$$\int_0^\infty \log I_0(y^{-\sigma})\,dy = \int_0^\infty F(x)x^{-1/\sigma}\,dx\,.$$

Note that $F(x) = 1 + O(x^{-1})$ as $x \to \infty$ and $F(x) = \frac{1}{2}x + O(x^3)$ as $x \to 0$. This shows that

$$\int_0^\infty F(x)x^{-1/\sigma}\,dx = \frac{\sigma}{1-\sigma} - \log K + o(1)$$

as $\sigma \to 1-$, with

$$-\log K = \int_0^1 F(x)\,x^{-1}\,dx + \int_1^\infty (F(x)-1)\,x^{-1}\,dx.$$

It then follows that $A(\sigma) \sim K\,(1-\sigma)$ as $\sigma \to 1-$. Also, as $\sigma \to \frac{1}{2}+$, we have

$$\int_0^\infty F(x)\,x^{-1/\sigma}\,dx = \int_0^1 F(x)\,x^{-1/\sigma}\,dx + O(1) \sim \int_0^1 \frac{1}{2}x^{1-1/\sigma}\,dx \sim \frac{1}{4(2\sigma-1)}\,.$$

It therefore follows that $A(\sigma) \sim 2(2\sigma - 1)$ as $\sigma \to \frac{1}{2}+$.

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