Anisotropic random walks and the asymptotically one-dimensional diffusions on the *abc*-gaskets

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Abstract

Asymptotically one-dimensional diffusion processes are studied on the class of fractals called *abc*-gaskets. The class is a set of certain variants of the Sierpiński gasket containing infinitely many fractals without any non-degenerate fixed point of renormalization maps. While the "standard" method constructing diffusions on the Sierpiński gasket and on nested fractals relies on the existence of a non-degenerate fixed point and hence it is not applicable to all *abc*-gaskets, the asymptotically one-dimensional diffusion is constructed on any *abc*-gasket by means of an unstable degenerate fixed point. To this end, the generating functions for numbers of steps of anisotropic random walks on the *abc*-gaskets are analyzed, according to the line of authors' previous studies. In addition, a general strategy of handling random walk sequences with more than one parameters for the construction of asymptotically one-dimensional diffusion is proposed.

Key words: diffusion process, random walk, finitely ramified fractal, branching process, renormalization group.

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1 Introduction

The purpose of this paper is to describe the essential part of the construction of asymptotically one-dimensional diffusions on the class of fractals called *abc*-gaskets[8], according to the program of [9]. Our conclusion is Theorem 2.2. The result was announced in [9] without proof.

A diffusion process on a fractal G can be viewed as a "continuum limit" of random walks on appropriate lattices on G by the following procedure: firstly choose lattices $G_N, N \in \mathbf{N}$, on G so that G_N monotonically "converges" to G: $G_1 \subset G_2 \subset \cdots \to G$; Secondly consider a random walk on each G_N satisfying the "consistency condition" (see (2.5) and (2.6)) which states that the random walk on G_N is a coarse-grained walk of the random walk on G_{N+1}; and then show the "convergence" of the sequence of random walks to a stochastic process on G called a diffusion process. In this procedure, the renormalization map plays an essential role, which yields the consistency condition between random walks on successive lattices G_N and G_{N+1}.

Our interest is diffusion processes on fractals. Standard constructions of diffusion processes on finitely ramified fractals, such as the Sierpiński gasket [15, 4] or the nested fractals [17], are essentially based on the existence of *non-degenerate fixed points* of renormalization maps (for the terminology, see Section 2). At first sight, the standard method might seem to be generalized to any simple variants of the Sierpiński gasket. In fact there are many examples of finitely ramified fractals that do not admit the standard construction of diffusion processes because of the absence of non-degenerate fixed points. Such examples are found in the class of *abc*-gaskets introduced in [8].

In this paper, we study a quite different type of diffusion processes on *abc*-gaskets called *asymptotically one-dimensional diffusions* based on the existence of *unstable degenerate fixed points* of renormalization maps (for the terminology, see Section 2). This concept was introduced in [9, 10] for the diffusion on Sierpiński gasket, and generalized for scale-irregular fractals [7], and pursued on Sierpiński carpet in the context of resistor networks [2, 3].

In what follows, we study the asymptotically one-dimensional diffusions on *abc*-gaskets along the line of [9] where this work was announced.

After this work was completed, there appeared the following works related to the same problems. Firstly, general convergence results for branching processes were given in [11], which would substitute the results in [10] (and perhaps may even simplify some arguments in Section 4). Secondly, an alternative construction of the asymptotically one-dimensional (lower dimensional) diffusions on a subclass of nested fractals was studied [5] by means of a general theory which relates the construction of diffusion processes to that of Dirichlet forms [6, 16, 12, 13]. Although their class of fractals does not cover all the *abc*-gaskets, some detailed asymptotic estimates on the heat kernels $p_t(x, x)$ are obtained and the homogenization problems are considered. Their work [5], based on the theory of Dirichlet forms, would simplify the analysis of F in Proposition 4.2 below and substitute the argument using the theory of branching processes. See also remarks at the end of Section 4. Finally, we notice that a characterization of asymptotically one-dimensional diffusions on the Sierpiński gasket by the exit distributions was given in [18].

This paper is organized as follows. In Section 2, we give the definition of *abc*-gaskets, formulate the problem, and state the result. The proof is given in the subsequent sections. In Section 3, we give the condition that a degenerate fixed point of the renormalization map is unstable and choose a sequence of anisotropic random walks on *abc*-gaskets keeping the consistency which guarantees the construction of the asymptotically one-dimensional diffusion. In Section 4, we give the estimate on the generating function for numbers of steps of anisotropic random walks on the *abc*-gaskets that are sufficient for the construction of the asymptotically one-dimensional diffusion along the line of [9]. The algebraic part of our proof of Proposition 4.2 is computer-aided because of the routine of rather lengthy calculations. In Appendix, a general strategy for the choice of sequence of random walks with a multi-parameter are given and an open problem is proposed.

2 Model and result

The *abc*-gaskets.

The *abc*-gasket was introduced in [8] (by an intrinsic definition). Here we will give another, intuitive, definition. Let us place, on a plane, a triangle $\triangle_0$ whose sides are of unit length. Let a, b, and c be positive integers, and put $a+b+c$ smaller triangles $\triangle_i$, $i = 1, 2, \cdots, a+b+c$ in $\triangle_0$ as in fig. 1, so that each $\triangle_i$ shares one or two sides in common with $\triangle_0$, and that no two small triangles have points in common except possibly for the vertices on the edges of $\triangle_0$. The small triangles are numbered so that the triangles $\triangle_i$, $i = 1, 2, \cdots, a+1$, have their horizontal edges on the horizontal edge of $\triangle_0$; the triangles $\triangle_i$, $i = a+1, a+2, \cdots, a+b+1$, have their right edges on the right edge of $\triangle_0$; the triangles $\triangle_i$, $i = a+b+1, a+b+2, \cdots, a+b+c, 1$, have their left edges on the left edge of $\triangle_0$. The sizes of the triangles are irrelevant for our subsequent discussion.

Next, consider the affine map φ_i, $i = 1, 2, \cdots, a+b+c$, which respectively maps $\triangle_0$ onto $\triangle_i$. We extend φ_1 to the whole plane as an affine map and denote it by the same symbol φ_1. We denote the inverse of the map by ${\varphi_1}^{-1}$. Let $\tilde{H}_0 = \triangle_0$, as the union of three line segments, and define $\tilde{H}_{n+1}$, $n = 1, 2, \cdots$, inductively by

$$\tilde{H}_{n+1} = {\varphi_1}^{-n-1} \left(\bigcup_{i=1}^{a+b+c} \varphi_i \circ {\varphi_1}^n(\tilde{H}_n) \right), \quad n = 0, 1, 2, \cdots, \tag{2.1}$$

and define $\tilde{H}_\infty = \bigcup_{n=0}^{\infty} \tilde{H}_n$.

Note that, by definition, $\tilde{H}_0 \subset \tilde{H}_1 \subset \tilde{H}_2 \subset \cdots$. Figures of first two constructions $\tilde{H}_1$ and $\tilde{H}_2$ are given in figures 1 and 2, respectively. As n is increased, the figure $\tilde{H}_n$ extends outwards, with the smallest structure being fixed (the scales of figure 1 and of 2 are different). The role of φ_1 playing special part in (2.1) means that $\tilde{H}_n$ extends in the right (and upward) direction, and that the left vertex of $\triangle_0$ remains as the leftmost vertex of $\tilde{H}_n$ for all n, hence it is also the leftmost vertex of $\tilde{H}_\infty$, which we denote by O. All the vertices of $\tilde{H}_\infty$ except O have four neighbor points. (By a neighbor point we mean a vertex connected by a line segment.) We can let O also have four neighbor points by defining $\tilde{H}'_\infty$ analogously; $\tilde{H}'_\infty = \bigcup_{n=0}^{\infty} \tilde{H}'_n$, where $\tilde{H}'_n$ is defined recursively by

$$\tilde{H}'_{n+1} = {\varphi_{a+1}}^{-n-1} \left(\bigcup_{i=1}^{a+b+c} \varphi_i \circ {\varphi_{a+1}}^n(\tilde{H}_n) \right), \quad n = 0, 1, 2, \cdots. \tag{2.2}$$

The rightmost vertex of $\tilde{H}_n$ for all n is the same, which we denote by O'. A pre-*abc*-gasket (of scale N) H_N is then defined by

$$H_N = {\varphi_1}^N \left(\tilde{H}_\infty \right) \cup T \left({\varphi_{a+1}}^N \left(\tilde{H}'_\infty \right) \right), \tag{2.3}$$

where T denotes a translation on the plane such that $T(O') = O$. (Actually, this procedure of doubling the figure makes all the vertices including O have similar structures of the two smallest blocks containing the vertex. We do this for technical simplicity in the analysis of random walks.) We denote the vertices of H_N by G_N.

The *abc*-gasket G is defined to be the closure of $\bigcup_{n=0}^{\infty} H_n$. (We take the closure so that the *abc*-gasket become a complete metric space.) G has infinitely small structures, whereas H_N has a non-zero smallest structure (specified by the scale N). Each smallest scale triangle of H_N to the right of O has a representation ${\varphi_1}^{N-n} \circ \varphi_{i_n} \circ \varphi_{i_{n-1}} \circ \cdots \circ \varphi_1(\triangle_0)$, for some non-negative integer n and a set of positive integers $i_1 \cdots i_n$, and a similar representation also for the smallest scale triangles to the left of O. We call an intersection of the *abc*-gasket G and the interior of one of such triangles a *block* of scale N.

In case of $a = b = c = 1$ the above construction coincides with that of the Sierpiński gasket.

Random walk on the pre-*abc*-gasket.

We consider random walks with nearest neighbor jumps on H_n. Probability laws of random walks on H_n are specified by assigning transition probabilities to each bond (edge) of H_n.

To this end, we first assign a conductivity $g(b)$ to a non-oriented bond b as follows. Let η and ζ be positive constants. If b is a horizontal bond, $g(b) = 1$. If b is a bond connecting upper left and lower right vertices, $g(b) = \eta$. If b is a bond connecting upper right and lower left vertices, $g(b) = \zeta$. Next we define a relative probability $\tilde{p}(\boldsymbol{b})$ of transition along a directed bond $\boldsymbol{b}$ to be $g(b)$, where $\boldsymbol{b}$ denotes a bond b with direction. Lastly we normalize the $\tilde{p}(\boldsymbol{b})$ and obtain a transition probability $p(\boldsymbol{b})$ so that the sum of $p(\boldsymbol{b})$'s for directed bonds $\boldsymbol{b}$ emerging from a vertex is equal to 1. The constants η and ζ parametrize the anisotropy between horizontal and off-horizontal directions.

For convenience' sake, we classify all the vertices of pre-*abc*-gasket H_n into 6 groups: types A, B, C, D, E, and F, respectively, according to the possible directions of bonds emerging from each vertex. (see fig. 2 and fig. 3.) In fig. 2, examples are shown for vertices belonging to the six groups A, B, C, D, E, and F, respectively. As is shown in fig. 3, a vertex of A, B, or C type has four bonds, of which directions are referred to as $p,\ q,\ r$, and s, respectively, while a vertex of D, E, or F type has two bonds (e.g. the directions of the two bonds emerging from a vertex of D type are referred to as p, q.) As a result, all the directed bonds of the pre-*abc*-gasket are classified into 18 types which we denote by A_p, A_q, $\cdots$, F_r, respectively. The transition probabilities assigned to them are shown in Table 1. If $a = 1$, the type D does not exist. For simplicity, we assume $a > 1$, $b > 1$, and $c > 1$. (A reader who is interested in the case $a = 1$ etc. should neglect irrelevant statements and formulae below.)

The renormalization map.

We next consider the consistency condition between random walks on H_n and on H_{n-1}. Let us fix a vertex of H_{n-1} and cut the block(s) of H_n to which the vertex belongs. To be specific, we assume that the vertex is of the type A. The vertex A belongs to two blocks (fig. 4). We denote the union of these blocks by K and the vertices of H_{n-1} contained in K by $A,\ A'_p,\ A'_q,\ A'_r$, and A'_s. Let $\Omega(A, t),\ t = p,\ q,\ r,\ s$, be the set of all walks on K

starting at the vertex A without reaching to any one of the vertices A'_p, A'_q, A'_r, and A'_s before the end at A'_t. Similarly we define the set $\Omega(X,t)$ of walks for $X = B,\ C,\ \cdots,\ F$, and for $t = p,\ q,\ r,\ s$, such that bonds of X_t type exist. Put

$$\Omega(X) = \bigcup_{t=p,q,r,s} \Omega(X,t),\ \ X = A, B, \cdots, F. \tag{2.4}$$

We assign a probability to each walk in $\Omega(X)$ by making a product of the transition probabilities set in Table 1. The probability measure on $\Omega(X)$ defined as above is denoted by $P_{X,\eta,\zeta}$.

Proposition 2.1. *Let $X = A,\ B,\ \cdots,\ F$, and let $t = p,\ q,\ r,\ s$, such that bonds of X_t type exist. The probability $P_{X,\eta,\zeta}(\Omega(X,t))$ is equal to the transition probability assigned to X_t given by Table 1 with η and ζ replaced by*

$$Y(\eta,\zeta) = \frac{(ac+a+c)\eta^2 + (ac+a+b)\eta\zeta + (ac+b+c)\eta + ac\zeta}{X(\eta,\zeta)}, \tag{2.5}$$

$$Z(\eta,\zeta) = \frac{(ab+a+b)\zeta^2 + (ab+a+c)\eta\zeta + (ab+b+c)\zeta + ab\eta}{X(\eta,\zeta)}, \tag{2.6}$$

respectively, where

$$X(\eta,\zeta) = bc\eta\zeta + (bc+a+c)\eta + (bc+a+b)\zeta + bc + b + c$$

The proof of this proposition will be sketched in Section 4. We call the set of equations (2.5) and (2.6), the *renormalization map.*

A random walk on the lattice H_N with $(\eta,\zeta) = (\eta_N,\zeta_N)$ in Table 1 and a random walk on the coarser lattice H_{N-1} with $(\eta,\zeta) = (\eta_{N-1},\zeta_{N-1})$ are consistent (i.e. the latter is a coarse-grained walk of the former), if the following relations hold:

$$\eta_{N-1} = Y(\eta_N,\zeta_N), \tag{2.7}$$

$$\zeta_{N-1} = Z(\eta_N,\zeta_N). \tag{2.8}$$

From random walks to diffusion.

In order to obtain a diffusion on the *abc*-gasket G, we must take the limit $N \to \infty$ of a consistent sequence of random walks on H_N satisfying (2.7) and (2.8). The simplest choice of the random walks is to put $(\eta_N,\zeta_N) = (\eta_*,\zeta_*)$, $N \in \mathbf{N}$, where (η_*,ζ_*) is a fixed point of the renormalization map (2.5)-(2.6). The standard method to construct a diffusion on Sierpiński gasket or a nested fractal belongs to this picture, where the fixed point is assumed to be *non-degenerate*, i.e.

$$\eta_*, \zeta_* \in (0,\infty). \tag{2.9}$$

Note that, if the fixed point is *degenerate*, namely

$$\eta_* = \zeta_* = 0, \text{ or } \eta_* < \zeta_* = \infty, \text{ or } \zeta_* < \eta_* = \infty \tag{2.10}$$

holds, the random walker can move in only one direction and hence, a sample path of the resulting diffusion is almost surely bound in a single line.

Asymptotically one-dimensional diffusion.

As is shown in Proposition 3.1, the renormalization map (2.5)-(2.6) has a non-degenerate fixed point, if and only if

$$a^{-1} < b^{-1} + c^{-1}, b^{-1} < c^{-1} + a^{-1}, c^{-1} < a^{-1} + b^{-1} \tag{2.11}$$

hold. Therefore, for *abc*-gaskets without (2.11), the standard method fails to construct non-degenerate diffusions. Including such cases, we can construct a quite different type of diffusion on any *abc*-gasket which we call an asymptotically one-dimensional diffusion.

In Proposition 3.2 we show that there exists a trajectory $(\eta_N, \zeta_N), N \in \mathbf{N}$, of renormalization map satisfying

$$\eta_N, \zeta_N \;>\; 0,\; N \in \mathbf{N}, \tag{2.12}$$

$$\lim_{N\to\infty} (\eta_N, \zeta_N) \;=\; (0,0), \tag{2.13}$$

if

$$a^{-1} < b^{-1} + c^{-1} \tag{2.14}$$

holds. (Note that the direction of increasing N is the inverse direction of the renormalization map.) The conditions (2.12) and (2.13) imply that the degenerate fixed point $(0,0)$ of the renormalization map (2.5)-(2.6) is unstable. The asymptotically one-dimensional diffusion process is constructed as the limit of random walks on H_N with $(\eta, \zeta) = (\eta_N, \zeta_N)$ in Table 1.

We state our main result of the present paper.

Theorem 2.2. *Under the assumption (2.14), there exists a continuous, non-constant, non-degenerate strong Markov process X_t, $t > 0$ on the abc-gasket such that*

(1) X_t is symmetric with respect to the Haussdorff measure which assigns mass $(a+b+c)^{-N}$ to each block of scale N;

(2) the transition semigroup P_t defined by $P_t f(x) = E^x f(X_t)$ maps the space of bounded continuous functions into itself;

(3) X_t is a weak limit of a sequence of random walks on $H_N, N = 1, 2, 3, \cdots$, with transition probabilities given by Table 1 with $(\eta, \zeta) = (\eta_N, \zeta_N)$ satisfying (2.12) and (2.13) and with the time unit $[(a+b)(a+b+c)]^{-N}$ (i.e. the time between succesive jumps on H_N).

We call the resulting process X_t in Theorem 2.2 *the asymptotically one-dimensional diffusion process.*

Remark. (1) Actual choices of η_N and ζ_N are given in Proposition 3.2. The choice of the Haussdorff measure is a natural extension of that for Sierpiński gasket [4, Lemma1.1].

(2) The assumtion (2.14) guarantees that the degenerate fixed point $(0,0)$ of the renormalization map (2.5)-(2.6) becomes unstable. In case (2.14) fails, then either $b^{-1} < c^{-1} + a^{-1}$ or $c^{-1} < a^{-1} + b^{-1}$ hold, so that at least one of the degenerate fixed points is always unstable, and by rotating the figure by 120^o (or 240^o, respectively), we can repeat the arguments in this paper to construct the diffusion process for any choice of a, b, c.

(3) One way to describe the asymptotically one-dimensional diffusion is as follows. Suppose that we are given a degenerate diffusion process on an *abc*-gasket. Let us perturb this situation by giving *infinitesimal* probabilities for off-horizontal jumps in a microscopic scale. Then we have two possibilities: either the perturbative effect survives or vanishes in the macroscopic scale according to whether the degenerate fixed point of the renormalization map is unstable or stable. The asymptotically one-dimensional diffusion belongs to the former picture.

(4) From the above theorem, we observe the tendency that the extreme anisotropy in the microscopic scale disappears in the macroscopic scale and the isotropy is gradually restored [9, 2, 3, 7, 14]. This phenomenon is not observed on regular spaces such as Euclidean spaces and smooth manifolds. It will be a special feature of fractal spaces of which mechanism may be clarified in an appropriate general framework. For a sketch of the mechanism of this phenomenon, see [2].

3 Transition probabilities

Fixed points

Let us study fixed points of the renormalization map (2.5)-(2.6):

(3.1) $$\eta = Y(\eta,\zeta) \ \ , \ \zeta = Z(\eta,\zeta) \ \ ,$$

It is convenient to consider the homogeneous equations corresponding to (2.5) and (2.6):

(3.2) $$X(\xi,\eta,\zeta) = bc(\xi+\eta)(\xi+\zeta)+\theta\xi,$$

(3.3) $$Y(\xi,\eta,\zeta) = ca(\eta+\xi)(\eta+\zeta)+\theta\eta,$$

(3.4) $$Z(\xi,\eta,\zeta) = ab(\zeta+\xi)(\zeta+\eta)+\theta\zeta,$$

where

(3.5) $$\theta = a(\eta+\zeta)+b(\zeta+\xi)+c(\xi+\eta).$$

Note that

(3.6) $$Y(\eta,\zeta) = \frac{Y(1,\eta,\zeta)}{X(1,\eta,\zeta)} \ \ , \ Z(\eta,\zeta) = \frac{Z(1,\eta,\zeta)}{X(1,\eta,\zeta)} \ \ .$$

Then the fixed point equation (3.1) is written as

(3.7) $$\xi:\eta:\zeta = X(\xi,\eta,\zeta):Y(\xi,\eta,\zeta):Z(\xi,\eta,\zeta).$$

This has trivial solutions

(3.8) $$\xi:\eta:\zeta = 1:0:0 \ , \ 0:1:0 \ , \ 0:0:1 \ ,$$

which give degenerate random walks.

Proposition 3.1. *The necessary and sufficient condition that (3.7) has a nontrivial (positive) solution is*

$$a^{-1} < b^{-1} + c^{-1}, b^{-1} < c^{-1} + a^{-1}, c^{-1} < a^{-1} + b^{-1}. \tag{3.9}$$

Moreover the nontrivial solution is given by

$$\xi : \eta : \zeta = \frac{1}{-a^{-1} + b^{-1} + c^{-1}} : \frac{1}{a^{-1} - b^{-1} + c^{-1}} : \frac{1}{a^{-1} + b^{-1} - c^{-1}}. \tag{3.10}$$

We omit the proof because it is easy and the result is not used later. It is also an easy exercise to show that the above nontrivial solution is a stable fixed point.

On the other hand, the trivial fixed points (3.8) may be stable or unstable. In fact, the proof of Proposition 3.2 below implies that the solution $\xi : \eta : \zeta = 1 : 0 : 0$ to (3.7), i.e., the solution $(\eta, \zeta) = (0, 0)$ to (3.1) is unstable if

$$a^{-1} < b^{-1} + c^{-1}. \tag{3.11}$$

Trajectory emerging from unstable degenerate fixed point.

In what follows, we assume (3.11) without loss of generality, since at least one of the inequalities of (3.9) must hold.

Proposition 3.2. *There exists two sequences η_n and ζ_n, $n = 1, 2, 3, \cdots$, of positive numbers which satisfy*

$$\begin{aligned} \eta_{n-1} &= Y(\eta_n, \zeta_n),\ n = 1, 2, \cdots, & (3.12)\\ \zeta_{n-1} &= Z(\eta_n, \zeta_n),\ n = 1, 2, \cdots, & (3.13)\\ C_1\delta^{-n} &< \eta_n < C_2\delta^{-n},\ n = 1, 2, \cdots, & (3.14)\\ C_3\delta^{-n} &< \zeta_n < C_4\delta^{-n},\ n = 1, 2, \cdots. & (3.15) \end{aligned}$$

Proof. Fixing $N \in \mathbf{Z}_+$, we consider sequences $\{\eta_n^{(N)} | 0 \le n \le N\}$ and $\{\zeta_n^{(N)} | 0 \le n \le N\}$ satisfying

$$\begin{aligned} \eta_{n-1}^{(N)} &= Y(\eta_n^{(N)}, \zeta_n^{(N)}),\ n = N, N-1, \cdots, 2, 1, & (3.16)\\ \zeta_{n-1}^{(N)} &= Z(\eta_n^{(N)}, \zeta_n^{(N)}),\ n = N, N-1, \cdots, 2, 1, & (3.17) \end{aligned}$$

and define random walks on H_n with transition probabilities given by Table 1 with $\eta = \eta_n^{(N)}$ and $\zeta = \zeta_n^{(N)}$. The recursion relations (3.16) and (3.17) are viewed as the consistency condition of probability laws of random walks on H_n and on H_{n-1}, which imply that the random walks on H_n are obtained by neglecting the fine structure of those on $H_m, m > n$.

Put

$$\delta = \frac{(a+1)(b+c)}{bc + b + c}. \tag{3.18}$$

Then, (3.11) implies $\delta > 1$.

Choose the initial values

$$\begin{aligned} \eta_N^{(N)} &= c\delta^{-N}\kappa, & (3.19)\\ \zeta_N^{(N)} &= b\delta^{-N}\kappa, & (3.20) \end{aligned}$$

for the recursion (3.16) and (3.17), where κ is an arbitrary positive number independent of N.

Since $\delta > 1$, $\eta_N^{(N)}$ and $\zeta_N^{(N)}$ are sufficiently small when N is large. On the other hand, if η and ζ are sufficiently small, (2.5) and (2.6) are written as

$$\begin{pmatrix} Y(\eta,\zeta) \\ Z(\eta,\zeta) \end{pmatrix} = R \begin{pmatrix} \eta \\ \zeta \end{pmatrix} + \begin{pmatrix} \tilde{Y}(\eta,\zeta) \\ \tilde{Z}(\eta,\zeta) \end{pmatrix}, \tag{3.21}$$

where

$$R = \frac{1}{bc+b+c} \begin{pmatrix} ac+b+c & ac \\ ab & ab+b+c \end{pmatrix}, \tag{3.22}$$

and the functions $\tilde{Y}(\eta,\zeta)$ and $\tilde{Z}(\eta,\zeta)$ obey the bounds

$$\begin{aligned} |\tilde{Y}(\eta,\zeta)| &< C_5(\eta^2+\zeta^2), \\ |\tilde{Z}(\eta,\zeta)| &< C_6(\eta^2+\zeta^2), \end{aligned}$$

for some constants independent of η and ζ. Note that the matrix R has eigenvalues $\delta (> 1)$ and $\dfrac{b+c}{bc+b+c} (< 1)$, with eigenvectors $\begin{pmatrix} c \\ b \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, respectively.

Then it is a routine work to show that the limits

$$\eta_n = \lim_{N\to\infty} \eta_n^{(N)} \tag{3.23}$$

$$\zeta_n = \lim_{N\to\infty} \zeta_n^{(N)} \tag{3.24}$$

exist and satisfy (3.12), (3.13), (3.14), (3.15) in Proposition 3.2. □

In the next section, we study the sequence of random walks on $H_n, n = 1, 2, 3, \cdots$ with transition probabilities given by Table 1 with $(\eta, \zeta) = (\eta_n, \zeta_n)$.

4 Hitting times

In our construction of the asymptotically one-dimensional diffusion process, the multi-distributions of numbers of steps of the random walks play the key role as is seen from the arguments for the Sierpiński gasket in [9, Section 2]. To this end, we analyze the generating functions for the numbers of steps of random walks on pre-*abc*-gaskets. A limit theorem related to the discrete-time multi-type non-stationary branching processes [10] is then applied, which, with arguments in [9], proves the existence of the asymptotically one-dimensional diffusions on the *abc*-gaskets. See [9] for details on the actual construction of the process.

Generating functions

We consider a random walk on H_N with transition probabilities given by Table 1 with $\eta = \eta_N$ and $\zeta = \zeta_N$, where η_N and ζ_N are defined in Proposition 3.2. In order to see the behavior of the walks in the scale of $H_n, n < N$, we generalize the set $\Omega(X, t)$ of walks in Section 3 as follows. Fix a vertex $X \in H_n$ of the type A and denote the "adjacent" vertices of X in H_n by $A''_p, A''_q, A''_r, A''_s$ in an analogous way as in fig. 4. In case (n, N) is $(n-1, n)$, the situation is exactly the same as fig. 4, but in general, the figure has a finer structure. Let $\Omega_{n,N}(X,t),\ t = p,\ q,\ r,\ s$, be the set of all walks starting at the vertex X without reaching to any one of the vertices $A''_p,\ A''_q,\ A''_r$, and A''_s before the end at A''_t. Similarly we define the set $\Omega_{n,N}(X,t)$ of walks for other types B, C, D, E, F of vertices X and for all directions $t = p, q, r, s$ such that bonds of X_t type exist. In particular, $\Omega(X,t) = \Omega_{n-1,n}(X,t)$. Put

(4.1) $$\Omega_{n,N}(X) = \bigcup_{t=p,q,r,s} \Omega_{n,N}(X,t),\ X = A, B, \cdots, F$$

and set the probability $P_{n,N,X}$ on $\Omega_{n,N}(X)$ by assigning to bonds of H_N the transition probabilities given by Table 1 with $\eta = \eta_N$ and $\zeta = \zeta_N$. Then, the probability

(4.2) $$\pi^{(n)}_{X_t} = P_{n,N,X}(\Omega_{n,N}(X,t))$$

is independent of N and is equal to the transition probability given by Table 1 with $\eta = \eta_n$ and $\zeta = \zeta_n$. We put

(4.3) $$\Pi_n = \mathrm{diag}(\pi^{(n)}_{A_p}, \pi^{(n)}_{A_q}, \cdots, \pi^{(n)}_{F_r}),$$

where $\mathrm{diag}(\alpha, \beta, \cdots, \gamma)$ stands for the diagonal matrix with diagonal elements $\alpha, \beta, \cdots, \gamma$.

For a walk $\omega \in \Omega_{n,N}(X)$ and for a set of 18 variables $\tilde{u} = (\tilde{u}_{A_p}, \tilde{u}_{A_q}, \cdots, \tilde{u}_{F_r})$, we use the following abbreviation

(4.4) $$\tilde{u}^{\omega} = \tilde{u}_{A_p}^{|\omega|_{A_p}} \tilde{u}_{A_q}^{|\omega|_{A_q}} \cdots \tilde{u}_{F_r}^{|\omega|_{F_r}},$$

where $|\omega|_{X_t}$ is the number of passes of the walk ω through bonds of X_t type. Then the generating function of numbers of steps is by definition

(4.5) $$\Phi_{n,N,X,t}(u) = \Pi_n^{-1} \sum_{\omega \in \Omega_{n,N}(X,t)} (\Pi_N u)^{\omega}, \quad u = (u_{A_p}, u_{A_q}, \cdots, u_{F_r})$$

and we write

(4.6) $$\Phi_{n,N}(u) = (\Phi_{n,N,A,p}(u), \Phi_{n,N,A,q}(u), \cdots, \Phi_{n,N,F,r}(u)).$$

Note that (3.12) and (3.13) imply

(4.7) $$\Phi_{n,N}(\Pi_N \vec{1}) = \vec{1},$$

where $\vec{1}$ denotes the vector with all elements 1.

Renormalization map for generating functions

The generating function has a recursive structure. To see this, we introduce the set of variables $\tilde{u} = (\tilde{u}_{A_p}, \tilde{u}_{A_q}, \cdots, \tilde{u}_{F_r})$ and define the mapping (independent of n)

(4.8) $$\tilde{U} = \tilde{F}(\tilde{u})$$

by putting

(4.9) $$\tilde{U}_{X_t} = \sum_{\omega \in \Omega(X,t)} \tilde{u}^{\omega}$$

(4.10) $$\tilde{U} = (\tilde{U}_{A_p}, \tilde{U}_{A_q}, \cdots, \tilde{U}_{F_r}).$$

Note that (3.12) and (3.13) imply

$$\Pi_{n-1}\vec{1} = \tilde{F}(\Pi_n \vec{1}). \tag{4.11}$$

Then the set of generating functions is written as

$$\Phi_{n,N}(u) = \Pi_n^{-1}\tilde{F}^{N-n}(\Pi_N u), \tag{4.12}$$

and hence is decomposed into the product of the mappings

$$U = \Pi_{n-1}^{-1}\tilde{F}(\Pi_n u). \tag{4.13}$$

Reduction of variables

The following lemma is easily shown.

Lemma 4.1. *If the variable $\tilde{u}$ satisfy the 'consistency conditions'*

$$\tilde{u}_{A_s} = \tilde{u}_{A_r}, \quad \tilde{u}_{B_s} = \tilde{u}_{B_p}, \quad \tilde{u}_{C_s} = \tilde{u}_{C_q}, \tag{4.14}$$

$$\tilde{u}_{A_p}\tilde{u}_{B_q} = \tilde{u}_{B_s}\tilde{u}_{A_q}, \quad \tilde{u}_{A_s}\tilde{u}_{B_q} = \tilde{u}_{B_r}\tilde{u}_{A_q}, \quad \tilde{u}_{A_p}\tilde{u}_{C_r} = \tilde{u}_{C_p}\tilde{u}_{A_r}, \tag{4.15}$$

$$\tilde{u}_{A_p}\tilde{u}_{C_s} = \tilde{u}_{C_p}\tilde{u}_{A_q}, \quad \tilde{u}_{D_q}\tilde{u}_{A_p} = \tilde{u}_{D_p}\tilde{u}_{A_q}, \quad \tilde{u}_{C_r}\tilde{u}_{F_p} = \tilde{u}_{C_p}\tilde{u}_{F_r}, \tag{4.16}$$

$$\tilde{u}_{A_p}\tilde{u}_{D_q}\tilde{u}_{E_r} = \tilde{u}_{A_r}\tilde{u}_{E_q}\tilde{u}_{D_p}, \tag{4.17}$$

then, $\tilde{U} = \tilde{F}(\tilde{u})$ also satisfy the same relations:

$$\tilde{U}_{A_s} = \tilde{U}_{A_r}, \quad \tilde{U}_{B_s} = \tilde{U}_{B_p}, \quad \tilde{U}_{C_s} = \tilde{U}_{C_q} \tag{4.18}$$

$$\tilde{U}_{A_p}\tilde{U}_{B_q} = \tilde{U}_{B_s}\tilde{U}_{A_q}, \quad \tilde{U}_{A_s}\tilde{U}_{B_q} = \tilde{U}_{B_r}\tilde{U}_{A_q}, \quad \tilde{U}_{A_p}\tilde{U}_{C_r} = \tilde{U}_{C_p}\tilde{U}_{A_r}, \tag{4.19}$$

$$\tilde{U}_{A_p}\tilde{U}_{C_s} = \tilde{U}_{C_p}\tilde{U}_{A_q}, \quad \tilde{U}_{D_q}\tilde{U}_{A_p} = \tilde{U}_{D_p}\tilde{U}_{A_q}, \quad \tilde{U}_{C_r}\tilde{U}_{F_p} = \tilde{U}_{C_p}\tilde{U}_{F_r}, \tag{4.20}$$

$$\tilde{U}_{A_p}\tilde{U}_{D_q}\tilde{U}_{E_r} = \tilde{U}_{A_r}\tilde{U}_{E_q}\tilde{U}_{D_p}. \tag{4.21}$$

In order to study the total number of steps of random walks, it suffices to analyze the function $\Pi_n^{-1}\tilde{F}^{N-n}(\Pi_N u)$ only for $u_{A_p} = u_{A_q} = \cdots = u_{F_r}$. In this case, $\tilde{u} = \Pi_n u$ satisfies the consistency conditions (4.14)–(4.17), since $\tilde{u} = \Pi_n \vec{1}$ satisfies (4.14)–(4.17). Then, as a result of the Lemma 4.1, we can reduce the 18 kinds of variables to 8 kinds. Our choice is the following:

$$z_1 = u_{A_r}, \; z_2 = u_{A_p}, \; z_3 = u_{A_q}, \; z_4 = u_{B_r}, \; z_5 = u_{C_r},$$
$$z_6 = u_{A_p}u_{D_q} = u_{D_p}u_{A_q}, \; z_7 = u_{E_r}, \; z_8 = u_{F_r}.$$

Namely, using the mapping

$$\Lambda : \; u \mapsto (u_{A_r}, u_{A_p}, u_{A_q}, u_{B_r}, u_{C_r}, u_{A_p}u_{D_q}, u_{E_r}, u_{F_r}), \tag{4.22}$$

we put

$$z = \Lambda u\,. \tag{4.23}$$

Note that under the consistency conditions in Lemma 4.1 the mapping Λ can be inverted. Define the matrix

$$D_n = \mathrm{diag}\ \Big(\frac{1}{2+\eta_n+\zeta_n}, \frac{\eta_n}{2+\eta_n+\zeta_n}, \frac{\zeta_n}{2+\eta_n+\zeta_n}, \frac{1}{1+2\eta_n+\zeta_n}, \frac{1}{1+\eta_n+2\zeta_n}, \frac{\eta_n\zeta_n}{(2+\eta_n+\zeta_n)(\eta_n+\zeta_n)}, \frac{1}{1+\zeta_n}, \frac{1}{1+\eta_n}\Big). \tag{4.24}$$

Since

$$D_n\Lambda = \Lambda\Pi_n, \tag{4.25}$$

the relation (4.13) is written as

$$Z = D_{n-1}^{-1}F(D_n z)\,, \tag{4.26}$$

where

$$F = \Lambda\circ\tilde{F}\circ\Lambda^{-1}\,, \tag{4.27}$$

$$Z = \Lambda U\,. \tag{4.28}$$

As a result, we have

$$\Phi_{n,N}(u) = \Lambda^{-1}D_n^{-1}F^{N-n}(D_N\Lambda u)\,. \tag{4.29}$$

Analysis of the renormalization map

The analysis of the generating function $\Phi_{n,N}$ reduces to that of the *renormalization map* F. The following proposition is the technical core of our work.

Proposition 4.2. *The function $F = \Lambda\circ\tilde{F}\circ\Lambda^{-1}$ has the expression*

$$F(D_n\vec{1}+t) = D_{n-1}\vec{1} + A_n t + R_n(t)\,, \quad n\in\mathbf{Z}_+,\ \|t\| < C_7\delta^{-n}, \tag{4.30}$$

where A_n is an 8×8 matrix independent of t and R_n is a $\mathbf{C}^8$-valued function with the following properties:

(1) The function R_n is analytic on $\{t|\,\|t\| < C_8\delta^{-n}\}$ with the bound

$$\|R_n(t)\| \le C_9\delta^n\|t\|^2, \quad \|t\| < C_{10}\delta^{-n},\ n\in\mathbf{Z}_+, \tag{4.31}$$

for some positive constants C_9 and C_{10} independent of N and t;

(2) There exists a matrix A such that

$$\|A - A_n\| < C_{11}\delta^{-n}, \tag{4.32}$$

where C_{11} is a constant independent of n;

(3) The matrix A has eigenvalues

$$(a+1)(a+b+c),\ \frac{(a+1)(2bc+b+c)}{bc+b+c},$$
$$a+1,\ a+1,\ a+1,\ a+1,\ \frac{(a+1)(b+c)}{bc+b+c},\ \frac{b+c}{bc+b+c}.$$

The eight eigenvalues of A satisfy

$$\begin{aligned} (a+1)(a+b+c) &> \frac{(a+1)(2bc+b+c)}{bc+b+c} \\ > a+1 = a+1 = a+1 = a+1 &> \frac{(a+1)(b+c)}{bc+b+c} > \frac{b+c}{bc+b+c}. \end{aligned}$$

In particular, the largest eigenvalue $\ell = (a+1)(a+b+c)$ of A is simple. The eigenvalue $\delta = \dfrac{(a+1)(b+c)}{bc+b+c}$ appeared in Proposition 3.2.

Sketch of proofs of Proposition 2.1 and Proposition 4.2

Proposition 2.1 is the result of the explicit calculation of the right hand side of (4.11). The proof of Proposition 4.2 needs the first order Taylor expansion of the left hand side of (4.30) with the remainder estimate. In fact the complete proof of Proposition 4.2 is very long. As mentioned in the Introduction, the algebraic part of the proof is computer-aided. The output of computer amounts to about 9×10^5 Bytes. Then, it would not be worthwhile to describe the detail of the calculations but it is reasonable to clarify the logical structure of our procedure so that a reader in principle can reproduce our calculations in a straightforward (long) way.

We consider the 'interior' of a block of H_n and number the six vertices adjacent to 'boundary points' of the block as fig. 5. Let $\Omega_{ij},\ i,j = 1,2,\cdots,6$, be the set of all walks in the interior of the block starting at i and ending at j without reaching any one of the six vertices except for the start and the end. We define the matrix $T = T(\tilde{u}) = (T_{ij})$ by

$$T_{ij} = \sum_{\omega \in \Omega_{ij}} \tilde{u}^{\omega},\ \ i,j = 1,2,\cdots,6. \tag{4.33}$$

In particular, we have

$$T_{23} = \tilde{u}_{A_q},\ T_{32} = \tilde{u}_{B_q},\ \text{etc.} \tag{4.34}$$

It is also an easy exercise to show the following:

$$T_{11} = -\frac{\lambda^a - \mu^a}{\lambda^{a-1} - \mu^{a-1}} + 1 - \tilde{u}_{A_p}\tilde{u}_{D_p}\,, \tag{4.35}$$

$$T_{22} = -\frac{\lambda^a - \mu^a}{\lambda^{a-1} - \mu^{a-1}} + 1 - \tilde{u}_{A_q}\tilde{u}_{D_q}\,, \tag{4.36}$$

$$T_{23} = T_{32} = -\frac{\lambda - \mu}{\lambda^{a-1} - \mu^{a-1}}(\tilde{u}_{A_r} + \tilde{u}_{A_q}\tilde{u}_{D_p})^{a-1}, \tag{4.37}$$

where λ and μ are the roots of the quadratic equation

$$x^2 - (1 - \tilde{u}_{A_p}\tilde{u}_{D_p} - \tilde{u}_{A_q}\tilde{u}_{D_q})x + (\tilde{u}_{A_r} + \tilde{u}_{A_q}\tilde{u}_{D_p})^2 = 0\,. \tag{4.38}$$

If $\lambda = \mu \neq 0$, the fractions should be evaluated after reductions. The other elements of the matrix T are given similarly. Put

$$W = (I - T)^{-1}. \tag{4.39}$$

Then W_{ij} gives the sum of $\tilde{u}^\omega$ over the set of all walks starting at i and ending at j, if the sum converges.

Let us describe the program to calculate $\tilde{F}$ defined by (4.8). As an example, we consider $\tilde{U}_{A_p}$ in the right hand side of (4.10). Connecting two blocks as fig. 4, we obtain

$$\tilde{U}_{A_p} = \frac{1}{\Theta_A}(\tilde{u}_{A_p}(W_{34}\tilde{u}_{B_p} + W_{35}\tilde{u}_{C_q}) + \tilde{u}_{A_r}(W_{24}\tilde{u}_{B_p} + W_{25}\tilde{u}_{C_q})), \tag{4.40}$$

where

$$\begin{aligned}\Theta_A = 1 \quad &- \quad \tilde{u}_{A_p}(W_{33}\tilde{u}_{B_s} + W_{32}\tilde{u}_{A_s}) - \tilde{u}_{A_q}(W_{66}\tilde{u}_{C_q} + W_{61}\tilde{u}_{A_r}) \\ &- \quad \tilde{u}_{A_r}(W_{23}\tilde{u}_{B_s} + W_{22}\tilde{u}_{A_s}) - \tilde{u}_{A_s}(W_{16}\tilde{u}_{C_q} + W_{11}\tilde{u}_{A_r}).\end{aligned}$$

The other components of $\tilde{U} = \tilde{F}(\tilde{u})$ are given similarly.

Proof of Proposition 2.1. For $\tilde{u} = \Pi_n\vec{1}$, the equation (4.38) has a double root $\lambda = \mu = \tilde{u}_{A_r} + \tilde{u}_{A_q}\tilde{u}_{D_p}$. Noting this fact, we explicitly calculate $\tilde{F}(\Pi_n\vec{1})$ and obtain (4.11). □

Proof of Proposition 4.2. The first term of the right hand side of (4.30) is obtained from (4.27),(4.25) and (4.11) as follows:

$$F(D_n\vec{1}) = \Lambda \circ \tilde{F} \circ \Lambda^{-1}(D_n\vec{1}) = \Lambda \circ \tilde{F}(\Pi_n\vec{1}) = \Lambda(\Pi_{n-1}\vec{1}) = D_{n-1}\vec{1}.$$

In order to obtain the second term, i.e. the derivative of $\tilde{F}$ at $\tilde{u} = \Pi_n\vec{1}$, we used REDUCE program on computer. In view of the explicit form of A_n produced by REDUCE, we can show the statements on the matrices A_n and A. The bound on the third term is obtained by looking into the remainder term produced at each step on the way from $\tilde{u}$ to $\tilde{U}$ described above. □

Convergence to the diffusion process

In order to show the convergence of the sequence of random walks on H_N considered above in the limit of $N \to \infty$, we study the asymptotic behavior of the generating function $\Phi_{n,N}$ or equivalently that of F^{N-n} (see (4.29)) as $N \to \infty$ with a fixed n. To this end, we put

$$f^{(n,N)}(s) = F^{N-n}(D_N \exp(-\ell^{-N} D_N^{-1} s)) \, . \tag{4.41}$$

Theorem 4.3. *There exist positive constants ϵ and C_{12} such that, for every $n \in \mathbf{Z}_+$:*

(1) The function $f^{(n,N)}(s)$, $N \geq n$, is analytic on $\|s\| < \epsilon\,\ell^n\delta^{-n}$ and converges uniformly to an analytic function $f^{(n)}(s)$ on $\|s\| < \epsilon\,\ell^n\delta^{-n}$ as $N \to \infty$.

(2) The function $f^{(n)}(s)$ has the expression

$$f^{(n)}(s) = D_n \vec{1} - \ell^{-n} B_n \, s + r^{(n)}(s) \, , \ \ \|s\| < \epsilon \, \ell^n \delta^{-n} \, , \tag{4.42}$$

with the bound

$$\|r^{(n)}(s)\| \leq C_{12} \delta^n \ell^{-2n} \|s\|^2 \, , \ \ \|s\| < \epsilon \, \ell^n \delta^{-n} \, , \tag{4.43}$$

where

$$B_n = \lim_{N \to \infty} \ell^{-N+n} A_{n+1} A_{n+2} \cdots A_N . \tag{4.44}$$

Proof. Since the largest eigenvalue ℓ of A in Proposition 4.2 is simple, the function $F = \Lambda \circ \tilde{F} \circ \Lambda^{-1}$ is $(\{D_n, \ell\})$-regular. [For the terminology, see [10].] Then we can apply Theorem 2 in [10] to our F and obtain the theorem. □

The above theorem claims that the time unit of random walks on H_N should be scaled as l^{-N}. Under this scaling, the standard tightness argument applies. The argument goes exactly parallel with that for the Sierpiński gasket. [See [9] for detail. Theorem 4.3 above corresponds to Proposition 2.4 in [9].] Thus we arrive at Theorem 2.2.

Remark. In [5] it is proved, for a rather general situation, that $\ell = N R_G$. Here N (in our case) counts the number of blocks (triangles) in a traingle of one scale large, or in other words, $N = 2^{d_f}$, where d_f is the fractal dimension. R_G is the resistance exponent for the one dimensional chain (in our case). (See also [1] for the basic idea using 'Einstein relations' to obtain such formula.) In the present case of the *abc*-gasket, $N = a + b + c$ and $R_G = a + 1$, hence our formula is reproduced. (Other eigenvalues of A cannot be obtained by this method.)

The eigenvalue δ is denoted by β^{-1} in [5]. In [5, Assumption 4.3] it is assumed that $R_G \beta > 1$, which, in our notation, is $\ell > (a + b + c)\delta$. On the other hand, the corresponding assumptions for our method to work is embodied in the $(\{D_n, \ell\})$-regularity assumed in [10, Theorem 2]. In particular, the assumption on the scaling factors is $\ell > \delta$. (Other conditions of the $(\{D_n, \ell\})$-regularity refer to other eigenvalues of A and to estimates on remainder terms of F.) Since $a + b + c > 1$, it seems that we have milder conditions than that of [5]. (One possible explanation is that [5] uses resistance metric which works nicely with the Dirichlet form theories, which however does not correspond nicely with Euclidean metric when the spectral dimension (or its appropriate analog for the asymptotically one dimensional diffusions) is greater than 2.)

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Appendix

Choice of random walk sequence for the multi-parameter case

An important point in the choice of parameter sequences for the *abc*-gaskets in Proposition 3.2 is that we are dealing with the multi-parameter case. Namely, we are considering cases where random walks on a (finitely ramified) pre-fractal is specified by a set of more than one non-negative parameters. In the case of the *abc*-gaskets, two parameters η and ζ are introduced. Another example of the multi-parameter case is the snowflake fractal [17]. As regards the asymptotically one-dimensional (lower dimensional) diffusions, such cases have not been considered for other fractals than the *abc*-gaskets. We here propose a general strategy suggested by the choice in Proposition 3.2, which may be applicable to other fractals. We leave it as an open problem to find general conditions with which the following idea may be applicable to construct asymptotically lower dimensional diffusions on fractals.

Let $r \geq 1$ be the number of parameters. Let $G_0 \subset G_1 \subset G_2 \subset \cdots$ be a sequence of vertices of pre-fractals, and assume that for each $n \in \mathbf{Z}_+$ a random walk Z_n on G_n is specified by the non-negative parameters $p^{(n)} = {}^t(p_1^{(n)}, p_2^{(n)}, \cdots, p_r^{(n)})$, chosen in such a way that for $m < n$, the m-decimated random walk of Z_n is Z_m. As in (2.5) and (2.6), there exists an $\mathbf{R}^r$–valued rational function $\Gamma = {}^t(\Gamma_1, \Gamma_2, \cdots, \Gamma_r)$ in r variables such that

$$p^{(n)} = \Gamma(p^{(n+1)}), \quad n \in \mathbf{Z}_+\,. \tag{A.45}$$

Let $p^* = {}^t(p_1^*, \cdots, p_r^*)$ be a fixed point of Γ with non-negative elements:

$$\Gamma(p^*) = p^*\,, \quad p_i^* \geq 0,\ i = 1, 2, \cdots, r. \tag{A.46}$$

Denote by $J = J(p)$ an r-dimensional matrix whose (i, j) element $J_{i,j}$ is given by $J_{ij}(p) = \dfrac{\partial \Gamma_i}{\partial p_j}(p)$. Let δ be that eigenvalue of $J(p^*)$ which has the largest absolute value. Assume that δ uniquely exists and is real and positive. To construct an asymptotically lower dimensional diffusion around p^*, it is necessary that p^* is an unstable fixed point of Γ, which implies $\delta > 1$. Let q^* be an eigenvector of $J(p^*)$ corresponding to the eigenvalue δ.

An intuitive choice is $p^{(n)} \approx p^* + \delta^{-n} q^*$, but from technical point of view, it is nicer to proceed in the following way. For each $N \in \mathbf{Z}_+$, put $p^{(N,N)} = p^* + \delta^{-N} q^*$, and define $p^{(N,n)}$, $n = 0, 1, 2, \cdots, N$, inductively by

$$p^{(N,n-1)} = \Gamma(p^{(N,n)}), \quad n = N, N-1, \cdots, 1. \tag{A.47}$$

Assume that

$$\tilde{\Gamma}(p) \stackrel{\text{def}}{=} \Gamma(p) - p^* - J(p^*)(p - p^*) \tag{A.48}$$

satisfies

$$|\tilde{\Gamma}_i(p)| \leq C \sum_{j=1}^{r} (p_j - p_j^*)^2, \quad i = 1, 2, \cdots, r \tag{A.49}$$

for some positive constant C independent of p. Let $p^{(n)} = \lim_{N\to\infty} p^{(N,n)}$, $n \in \mathbf{Z}_+$. Then

$$p^{(n-1)} = \Gamma(p^{(n)}), \quad C_1 \delta^{-n} q_i^* \leq p_i^{(n)} - p_i^* < C_2 \delta^{-n} q_i^*, \quad i = 1, 2, \cdots, r,\ n \in \mathbf{Z}_+, \tag{A.50}$$

for some positive constants C_1 and C_2 independent of p. This provides a generalized form of the Proposition 3.2.

General conditions on the fractals, for which this idea works, is left as an open problem.

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	p	q	r	s
A	$\eta/(2+\eta+\zeta)$	$\zeta/(2+\eta+\zeta)$	$1/(2+\eta+\zeta)$	$1/(2+\eta+\zeta)$
B	$\eta/(1+2\eta+\zeta)$	$\zeta/(1+2\eta+\zeta)$	$1/(1+2\eta+\zeta)$	$\eta/(1+2\eta+\zeta)$
C	$\eta/(1+\eta+2\zeta)$	$\zeta/(1+\eta+2\zeta)$	$1/(1+\eta+2\zeta)$	$\zeta/(1+\eta+2\zeta)$
D	$\eta/(\eta+\zeta)$	$\zeta/(\eta+\zeta)$	0	0
E	0	$\zeta/(1+\zeta)$	$1/(1+\zeta)$	0
F	$\eta/(1+\eta)$	0	$1/(1+\eta)$	0

Table 1: Transition probabilities

Table caption.

Table 1 : Transition probabilities

Figure captions.

Figure 1: The block of pre-abc-gasket with $a = 2, b = 4, c = 3$.

Figure 2: The second stage construction of pre-abc-gasket with $a = 2, b = 4, c = 3$.

Figure 3: We classify vertices of a pre-abc gasket into 6 groups A, B, C, D, E, and F according to the possible directions of transitions from each vertex. We refer to the possible directions by the symbols p, q, r, and s.

Figure 4: The vertex $A \in H_{n-1}$ belongs to two blocks of H_n. We denote the vertices of H_{n-1} contained in the two blocks by A, A'_p, A'_q, A'_r, and A'_s.

Figure 5: The 'interior' of a block of H_n. We number the six vertices adjacent to 'boundary points' of the block as above.

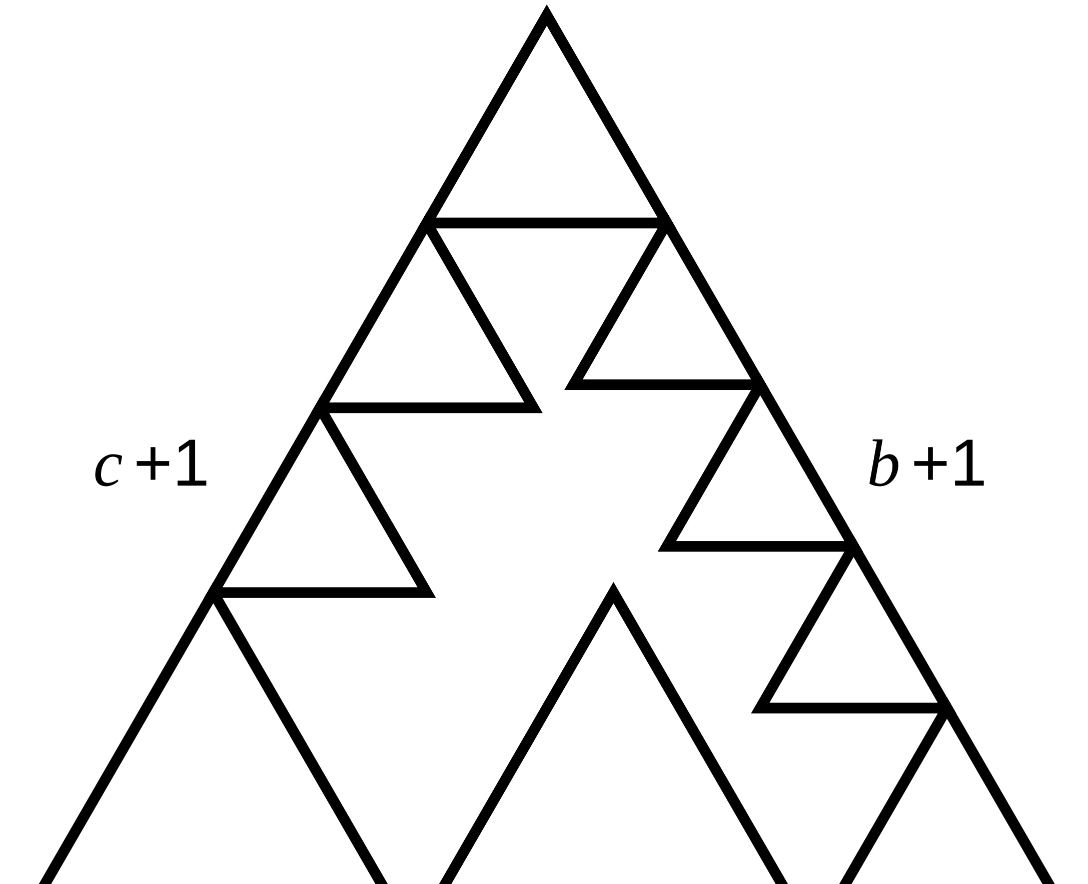

fig.1

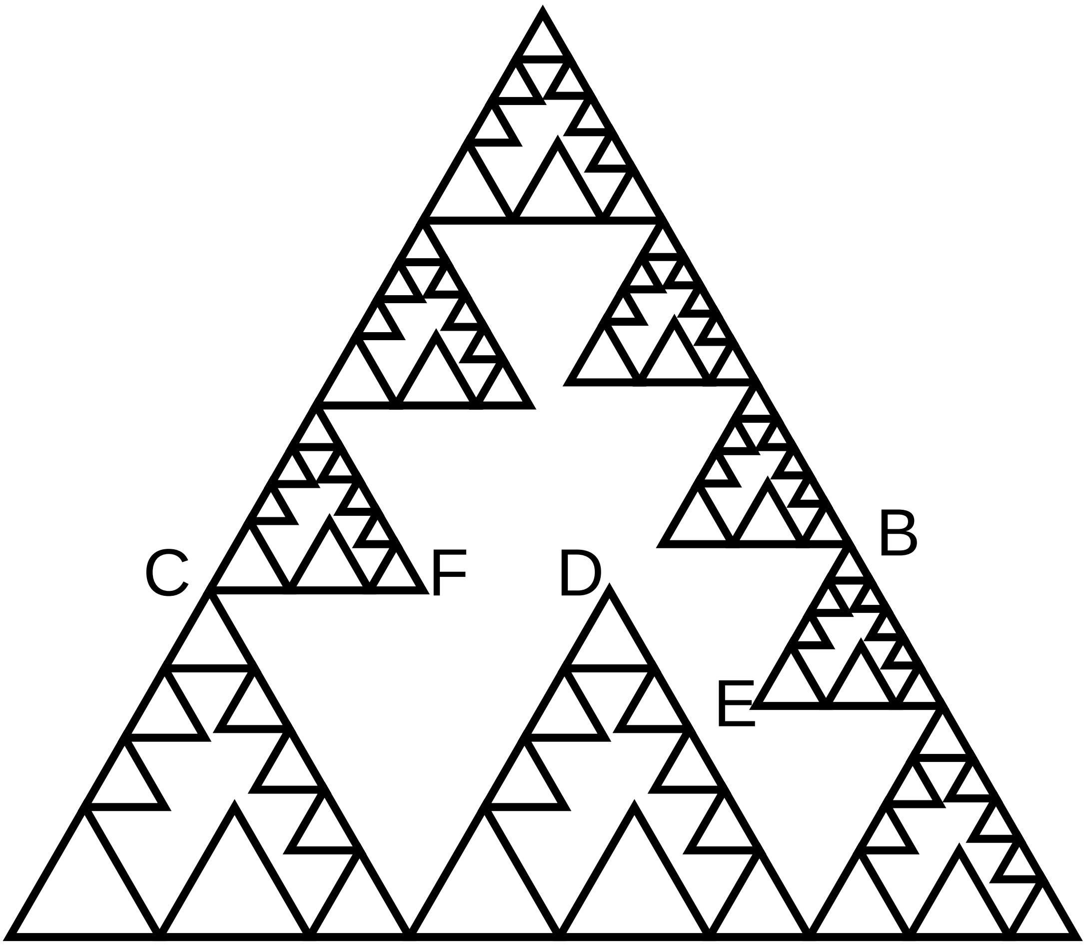

fig.2

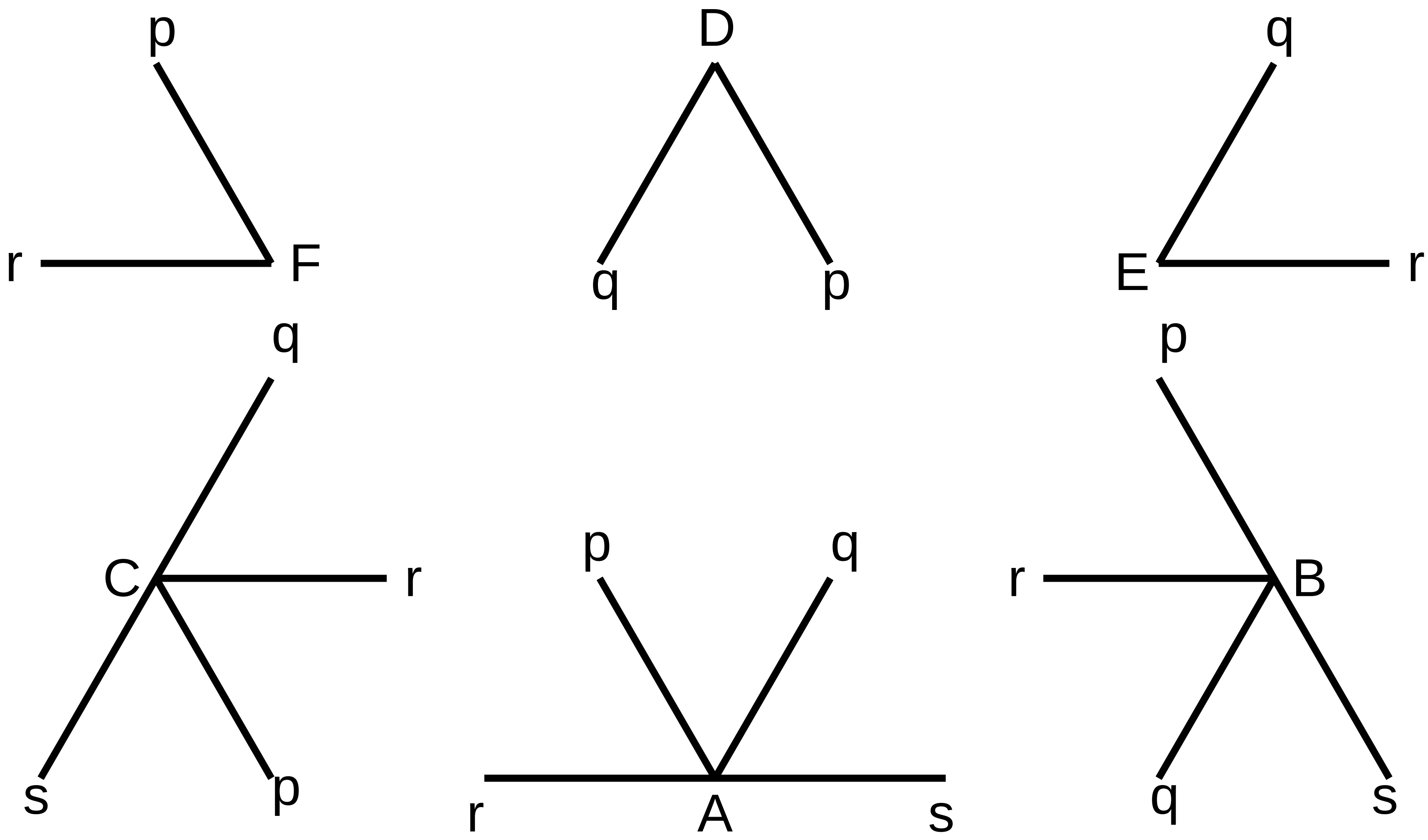

fig.3

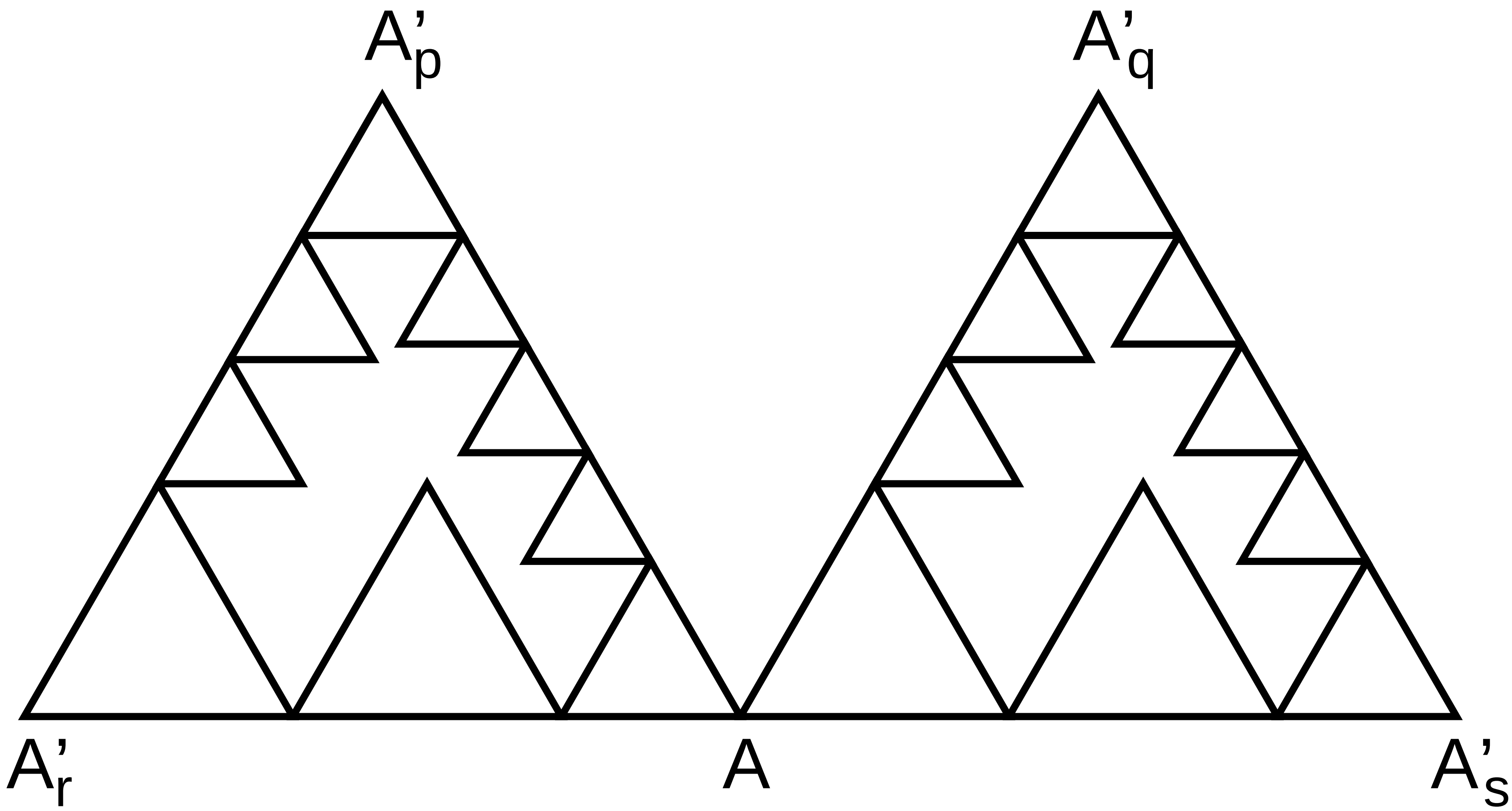

fig.4

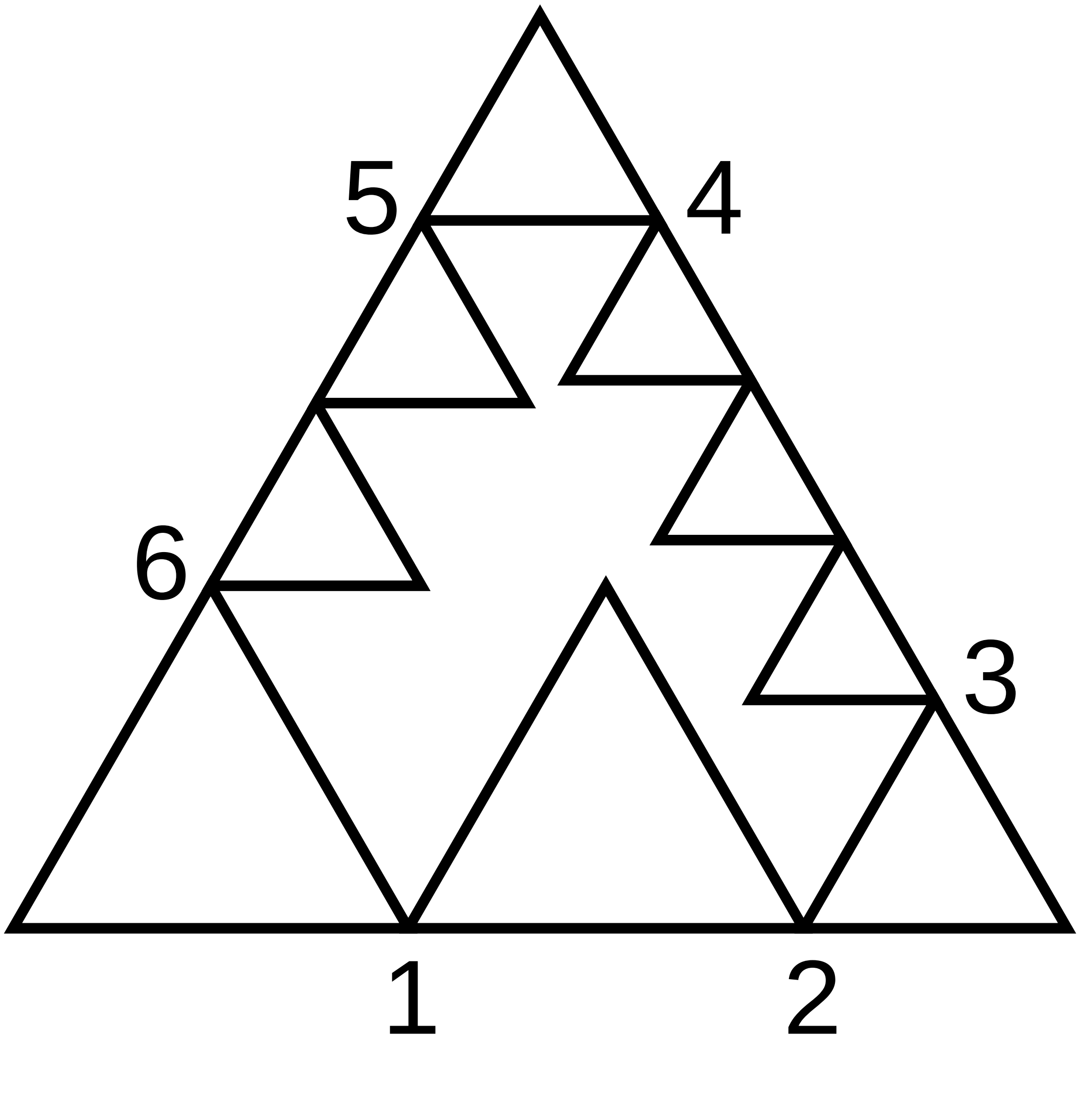

fig.5