Essential self-adjointness of Dirichlet operators on a path space with Gibbs measures via an SPDE approach

(joint work with Michael RÖCKNER)

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§1. Introduction (Problem)

- state space: infinite volume path space $C(\mathbb{R}, \mathbb{R}^d)$
- tangent space: $H := L^2(\mathbb{R}, \mathbb{R}^d)$
- underlying measure: Gibbs measure μ

associated with the (formal) Hamiltonian

$$H(w) := \frac{1}{2} \int_{\mathbb{R}} |w'(x)|^2_{\mathbb{R}^d} dx + \int_{\mathbb{R}} U(w(x)) dx,$$

where $U : \mathbb{R}^d \to \mathbb{R}$ is a self-interaction potential.

Heuristically, μ is given by

$$\mu(dw) = Z^{-1} e^{-H(w)} \prod_{x \in \mathbb{R}} dw(x).$$

Consider a (pre-)Dirichlet form

$$\mathcal{E}(F,G) := \frac{1}{2}\int \big(D_H F(w), D_H G(w)\big)_H \mu(dw)$$

for $F, G \in \mathcal{F}\mathcal{C}_b^\infty$ (smooth cylinder functions).

$\Longrightarrow$ We can consider a (pre-)Dirichlet operator $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$ through

$$\mathcal{E}(F,G) = -(\mathcal{L}_0 F, G)_{L^2(\mu)}.$$

Our problem: Is the pre-Dirichlet operator $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$ essential self-adjoint in $L^2(\mu)$?

• Related works for infinite-dimensional settings:

(i) • Takeda ('85), • Röckner-Zhang ('92),

• Shigekawa ('95) etc.

$\Rightarrow$ Functional analytic approach (e.g. Malliavin calculus)

under $\mu(dw) = \rho(w)\mathcal{W}(dw)$

(ii) • Albeverio-Kondratiev-Röckner ('95$\sim$) etc.

$\Rightarrow$ (Finite dimensional) approximation approach with stochastic analysis (stochastic flow)

(iii) • Da Prato (2000$\sim$), • Da Prato-Röckner (2002) etc.

$\Rightarrow$ SPDE approach

§2. Framework and Results

At the beginning, we introduce some notations and objects we will working with.

- weight function $\rho_r \in C^\infty(\mathbb{R}, \mathbb{R})$, $r \in \mathbb{R}$, is defined by $\rho_r(x) := e^{r\chi(x)}, x \in \mathbb{R}$, where χ is a convex even smooth function with $\chi(x) = |x|$ for $|x| \geq 1$. ($\rho_r(x) \approx e^{r|x|}$)
- $E := L^2(\mathbb{R}, \mathbb{R}^d; \rho_{-2r}(x)dx)$, ($r > 0$ fixed) with $(X, Y)_E := \int_{\mathbb{R}} (X(x), Y(x))_{\mathbb{R}^d} \rho_{-2r}(x)dx$.
- $H := L^2(\mathbb{R}, \mathbb{R}^d)$

Before giving a Gibbs measure, we impose some conditions on the potential function U.

(U1): $U \in C^1(\mathbb{R}^d, \mathbb{R})$ and $\exists K_1 \in \mathbb{R}$ s.t.

$$\left(\nabla U(z_1) - \nabla U(z_2), z_1 - z_2\right)_{\mathbb{R}^d} \geq -K_1 |z_1 - z_2|^2_{\mathbb{R}^d} \text{ for } z_1, z_2 \in \mathbb{R}^d.$$

(U2): $\exists K_2 > 0, \exists p > 0$ s.t.

$$|\nabla U(z)|_{\mathbb{R}^d} \leq K_2(1 + |z|^p_{\mathbb{R}^d}) \text{ for } z \in \mathbb{R}^d.$$

(U3): $\lim_{|z|_{\mathbb{R}^d} \to \infty} U(z) = \infty.$

Example: $U(z) = a(|z|^4_{\mathbb{R}^d} - |z|^2_{\mathbb{R}^d}), a > 0$

Under **(U1)** and **(U3)**, we can construct a Gibbs measure on $C(\mathbb{R}, \mathbb{R}^d)$ in the following manner:

- Consider a Schrödinger operator $H := -\frac{1}{2}\Delta + U$ on $L^2(\mathbb{R}^d, \mathbb{R})$. H has purely discrete spectrum and a complete set of eigenfunctions.

$\Rightarrow$ · $\lambda_0 (> \min U)$: the lowest eigenvalue of H,

· Ω : ground state of H with $\|\Omega\|_{L^2(\mu)} = 1$ and $\Omega > 0$.

i.e., $H\Omega = \lambda_0 \Omega$. $(e^{-tH}\Omega = e^{-t\lambda_0}\Omega)$

- $\mathcal{W}_{-T,z_1;T,z_2}$ $(T > 0, z_1, z_2 \in \mathbb{R}^d)$: pinned BM measure with

$$\mathcal{W}_{-T,z_1;T,z_2}(w(-T) = z_1, w(T) = z_2) = 1.$$

- $p(t, z_1, z_2)$: transition probability of d-dim standard BM.

- σ-fields of the space $C(\mathbb{R}, \mathbb{R}^d)$:

$$\mathcal{B} := \sigma(w(x); x \in \mathbb{R}),$$
$$\mathcal{B}_T := \sigma(w(x); -T \le x \le T),$$
$$\mathcal{B}_{T,c} := \sigma(w(x); x < -T, x > T).$$

$\Rightarrow$ We define a probability measure on $C(\mathbb{R}, \mathbb{R}^d)$ by

$$\mu(A) := e^{2T\lambda_0} \int_{\mathbb{R}^d} dz_1 \Omega(z_1) \int_{\mathbb{R}^d} dz_2 \Omega(z_2)$$
$$\times p(2T, z_1, z_2) \mathbb{E}^{\mathcal{W}_{-T,z_1;T,z_2}} \left[e^{-\int_{-T}^{T} U(w(x))dx}; A \right]$$

for $A \in \mathcal{B}_T$ and by extending the above to a measure on $\mathcal{B}$.

Remark: $p(2T, z_1, z_2) \mathbb{E}^{\mathcal{W}_{-T,z_1;T,z_2}} \left[e^{-\int_{-T}^{T} U(w(x))dx} \right]$ is equal to $e^{-2TH}(z_1, z_2)$. (Feynman-Kac formula)

Properties of μ ♣ We can obtain the estimate

$$\int \Big(\int_{\mathbb{R}} |w(x)|_{\mathbb{R}^d}^{2m} \rho_{-2r}(x) dx \Big) \mu(dw)$$

$$\leq \frac{1}{r} \int_{\mathbb{R}^d} |z|_{\mathbb{R}^d}^{2m} \Omega(z)^2 dz < \infty, \ m \in \mathbb{N}.$$

Then we notice that $\mu(\mathcal{C}) = 1$, where

$$\mathcal{C} := \bigcap_{r>0} \{ w \in C(\mathbb{R}, \mathbb{R}^d); \|w\|_{r,\infty} < \infty \}.$$

$$\Big(\ \|w\|_{r,\infty} := \sup_{x \in \mathbb{R}} |w(x)|_{\mathbb{R}^d} \rho_{-r}(x) \ \Big)$$

$\Rightarrow$ Since $\mathcal{C} \hookrightarrow E$ is continuous, we can regard μ as a probability measure on E.

♣ DLR-equation:

For $\forall T \in \mathbb{N}$, μ-a.e. $\xi \in C(\mathbb{R}, \mathbb{R}^d)$:

$$\mu(dw|\mathcal{B}_{T,c})(\xi) = Z_{T,\xi}^{-1} e^{-\int_{-T}^{T} U(w(x))dx} \times \mathcal{W}_{-T,\xi(-T);T,\xi(T)}(dw).$$

(Definition of Gibbs measures)

• Betz-Lörinczi ('03)$\cdots$ If $\exists a > 2, U(z)$ grows at infinity faster than $|z|_{\mathbb{R}^d}^{a}$ but slower than $|z|_{\mathbb{R}^d}^{2a-2}$
$\Longrightarrow$ there is a unique Gibbs measure on $C(\mathbb{R}, \mathbb{R}^d)$.

♣ Quasi-invariance:

For every $k \in C_0^\infty(\mathbb{R}, \mathbb{R}^d)$,

$$\mu \sim \mu(k + \cdot) \text{ and } \mu(k + dw) = \Lambda(k, w)\mu(dw),$$

where

$$\Lambda(k, w) = \exp\left\{ \int_{\mathbb{R}} \Big(U(w(x)) - U(w(x) + k(x)) - \frac{1}{2}|k'(x)|^2 + (w(x), \Delta_x k(x))_{\mathbb{R}^d} \Big) dx \right\}$$

and $\Delta_x := d^2/dx^2$.

- the space of smooth cylinder functions:

$$\mathcal{F}\mathcal{C}_b^\infty := \{F(w) = f(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle);$$
$$n \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^n, \mathbb{R}),$$
$$\varphi_1, \cdots, \varphi_n \in C_0^\infty(\mathbb{R}, \mathbb{R}^d)\},$$

where $\langle w, \varphi_i\rangle := \int_{\mathbb{R}} (w(x), \varphi_i(x))_{\mathbb{R}^d} dx, \ w \in E.$

♣ $\mathcal{F}\mathcal{C}_b^\infty \hookrightarrow L^2(\mu)$ (dense)

- H-Fréchet derivative $D_H F : E \to H$:

$$D_H F(w)(\cdot) := \sum_{i=1}^{n} \partial_i f(\langle w, \varphi_1\rangle, \cdots, \langle w, \varphi_n\rangle)\varphi_i(\cdot)$$

for $F \in \mathcal{F}\mathcal{C}_b^\infty$.

Define a (pre-)Dirichlet form $(\mathcal{E}, \mathcal{F}C_b^\infty)$ by

$$\mathcal{E}(F,G) := \frac{1}{2}\int_E \left(D_H F(w), D_H G(w)\right)_H \mu(dw)$$

for $F, G \in \mathcal{F}C_b^\infty$. By the quasi-invariance of μ, we obtain

$$\boxed{\mathcal{E}(F,G) = -(\mathcal{L}_0 F, G)_{L^2(\mu)}, \quad F, G \in \mathcal{F}C_b^\infty, \cdots (\dagger)}$$

where

$$\mathcal{L}_0 F = \frac{1}{2}\mathrm{Tr}(D_H^2 F(w)) + \frac{1}{2}\Big\{\langle w, \Delta_x D_H F(w(\cdot))\rangle - \langle \nabla U(w(\cdot)), D_H F(w)\rangle\Big\}$$

By (†), $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$ is dissipative on $L^2(\mu)$,

i.e., $(\mathcal{L}_0 F, F)_{L^2(\mu)} \leq 0$ for $F \in \mathcal{F}\mathcal{C}_b^\infty$.

$\Longrightarrow$ $\exists$ self-adjoint extension of $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$

(Freidrichs extension)

$\Updownarrow$

- $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$: the closure of $(\mathcal{E}, \mathcal{F}\mathcal{C}_b^\infty)$ w.r.t $\mathcal{E}_1^{1/2}$-norm

(Minimal Dirichlet form)

Theorem 1 (i) The pre-Dirichlet operator $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$ is essentially self-adjoint in $L^2(\mu)$, i.e., $(\overline{\mathcal{L}_2}, \mathrm{Dom}(\overline{\mathcal{L}_2}))$: closure of $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$ in $L^2(\mu)$ is self-adjoint.

(ii)

$$e^{t\overline{\mathcal{L}_2}} F(w) = P_t F(w), \quad \mu\text{-a.s. } w, \ F \in L^2(\mu),$$

where $\{P_t\}_{t\geq 0}$ is the transition semigroup corresponding to the parabolic SPDE

$$dX_t(x) = \frac{1}{2}\{\Delta_x X_t(x) - \nabla U(X_t(x))\}dt$$
$$+dB_t(x), \ \ x \in \mathbb{R}, \ t > 0, \cdots \text{(GL)}$$

where $\{B_t\}_{t\geq 0}$ is a H-cylindrical Brownian motion.

As a consequence of Theorem 1, we can also obtain the Markov uniqueness.

Theorem 2 (Markov uniqueness) The Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is the unique extension of $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$.

- $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$: Dirichlet form in $L^2(\mu)$ is an extension of $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$.

$\underset{\text{def}}{\Longleftrightarrow}$ · $\mathcal{F}\mathcal{C}_b^\infty \subset \mathrm{Dom}(\mathcal{E})$,

· $\mathcal{E}(F, G) = (-\mathcal{L}_0 F, G)_{L^2(\mu)}$ holds for $\forall F \in \mathcal{F}\mathcal{C}_b^\infty, \forall G \in \mathrm{Dom}(\mathcal{E})$.

• Albeverio-Kusuoka ('88),
• Albeverio-Kusuoka-Röckner ('90) etc.

$\Longrightarrow$ Characterization of the maximal Dirichlet form $(\mathcal{E}^+, \mathcal{D}(\mathcal{E}^+))$

Application (Rademacher type theorem)

$F : E \to \mathbb{R}$ measurable s.t. for $\forall w \in E$, $\forall h \in H$,

$$|F(w+h) - F(w)| \leq C\|h\|_H$$

$$\underset{\text{Kusuoka}}{\Longrightarrow} \quad F \in \mathcal{D}(\mathcal{E}^+) \quad \underset{\text{Theorem 2}}{\Longrightarrow} \quad F \in \mathcal{D}(\mathcal{E})$$

♣ If we consider "H-distance function", this plays a key role to give the upper bound of $(P_t 1_A, 1_B)_{L^2(\mu)}$.
(cf. K. : Potential Anal. (2005))

§3 Sketch of the Proof for the Main Theorem

Our approach is essentially based on Da Prato and Röckner's one (2002).

• $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$: dissipative ($\longrightarrow$ closable)

$\Rightarrow$ $(\overline{\mathcal{L}_2}, \mathrm{Dom}(\overline{\mathcal{L}_2}))$: closure of $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$

(dissipativity also holds.)

$\Downarrow$

<u>Aim:</u> $(\overline{\mathcal{L}_2}, \mathrm{Dom}(\overline{\mathcal{L}_2}))$: m-dissipative, i.e.,

$\exists\lambda > 0,\ \mathrm{Range}(\lambda - \overline{\mathcal{L}_2}) = L^2(\mu).$

$\Uparrow$ (Lumer-Phillips Theorem)

It is sufficient to show

$$\exists\lambda > 0, \mathcal{F}\mathcal{C}_b^\infty \subset \mathrm{Range}(\lambda - \overline{\mathcal{L}_2})(\subset L^2(\mu)).$$

Hence it is sufficient to show

$\exists \lambda > 0, \forall F \in \mathcal{FC}_b^\infty, \exists \Phi \in \mathrm{Dom}(\overline{\mathcal{L}_2})$ s.t.

$$\lambda \Phi - \overline{\mathcal{L}}_2 \Phi = F \cdots (\sharp)$$

(infinite-dimensional elliptic problem)

♣ Candidate:

$$\Phi = \int_0^\infty e^{-\lambda t} P_t F dt, \ \lambda > \frac{K_1}{2} + r^2$$

$$\Uparrow$$

Facts on the SPDE (GL) (Iwata, Funaki, . . .)

(i) SPDE (GL) has a unique (pathwise) solution $(X_t^w(\cdot))_{t\geq 0}$ living in $C([0,\infty),\mathcal{C})$ for an initial data $w \in \mathcal{C}$.

(ii) For $F \in \mathcal{F}\mathcal{C}_b^\infty$, we set

$$P_t F(w) := \mathbb{E}[F(X_t^w)], w \in \mathcal{C}, t \geq 0.$$

Then $(P_t)_{t\geq 0}$ can be regarded as a C_0-contraction symmetric semigroup on $L^2(\mu)$.

(iii) Its infinitesimal generator is an extension of the (pre-)Dirichlet operator $(\mathcal{L}_0, \mathcal{F}\mathcal{C}_b^\infty)$.

($\leftarrow$ an easy consequence of Itô's formula)

- Difficulty: It is difficult to show $\Phi \in \mathrm{Dom}(\overline{\mathcal{L}_2})$ directly!!

⇓ How to show?

We insert a tractable space which corresponds to the Ornstein-Uhlenbeck (OU-)operator. i.e., We want to understand as $\overline{\mathcal{L}_2}$ =(OU-operator)+(perturbation).

Formulation of the OU operator

Step 1. Take $\kappa > 0$ s.t. $\kappa > 2r^2$

$$(\longrightarrow \omega := \frac{\kappa}{2} - r^2 > 0)$$

Set $S_t w(x) := e^{-\kappa t/2} \int_{\mathbb{R}} g(t, x, y) w(y) dy$

$\Rightarrow$ $(S_t)_{t\geq 0}$: C_0-contraction semigroup on E.

(Note it is not symmetric!)

$\Rightarrow$ $(A, \mathrm{Dom}(A))$: infinitesimal generator of (S_t). $\left(A = \frac{1}{2}(\Delta_x - \kappa) \right)$

Step 2. Consider a parabolic SPDE

$$dY_t(x) = \frac{1}{2}\{\Delta_x Y_t(x) - \kappa Y_t(x))\}dt + dB_t(x), \quad x \in \mathbb{R}, \ t > 0 \quad \cdots \text{(OU)}$$

with an initial data $w \in E$.

$\Rightarrow$ We can write down the solution of (OU) as

$$\boxed{Y_t^w = S_t w + \int_0^t S_{t-s}\sqrt{Q}\,dW_s,\ t \geq 0,\ \cdots (\star)}$$

where

- $Q \in L(E, E)$: $Qw := \rho_{-2r} \cdot w$
- $(W_t)_{t\geq 0}$: E-cylindrical Brownian motion.

<u>Remark:</u> (mean of $(\star)$)$= S_t w$,

(covariance of $(\star)$)$= \int_0^t S^*_{t-s} Q S_{t-s} ds (=: Q_t)$

$\Rightarrow$ We easily see $Q_t : E \to E$ is a trace class operator.

⇒ Define the OU-semigroup $(R_t)_{t\geq 0}$ by

$$R_t F(w) := \mathbb{E}[F(Y_t^w)] = \int_E F(S_t w + y) N_{Q_t}(dy)$$

How should we choose a good domain for $(R_t)_{t\geq 0}$?

⇒ Da Prato, Pliola, Tubaro etc. introduced the following subspaces of $C(E)$:

- $UC_{b,2}(E)$ $\cdots$ the set of all functions $F : E \longrightarrow \mathbb{R}$ with $\frac{F(\cdot)}{1+\|\cdot\|_E^2}$ is uniformly continuous and bounded. This is a Banach space w.r.t the norm $\|F\|_{b,2} := \sup_{w\in E} \frac{|F(w)|}{1+\|w\|_E^2}$.

• $C^1_{b,2}(E)$ $\cdots$ the subspace of $UC_{b,2}(E)$ of those functions F are continuously differentiable with $\|DF\|_{b,2} := \sup_{w \in E} \frac{\|DF(w)\|_E}{1+\|w\|_E^2} < \infty$, where $DF : E \to E$ is the E-Fréchet derivative of F.

Remark: $D_H F = QDF$

$\Longrightarrow$ $(R_t)_{t \geq 0}$: semigroup on $UC_{b,2}(E)$

♣ It is not strongly continuous! But it is regarded as a π-semigroup in the sense of Da Prato and Priola.

Step 3. Define the OU-operator L through the resolvent

$$
\begin{aligned}
(\lambda - L)^{-1} F(w) &= \Psi_\lambda F(w) \\
&:= \int_0^\infty e^{-\lambda t} R_t F(w)\, dt, \ \lambda > 0, w \in E,
\end{aligned}
$$

and set

- $\mathcal{D}(L; UC_{b,2}(E)) := \Psi_\lambda(UC_{b,2}(E))$,
- $\mathcal{D}(L; C^1_{b,2}(E)) := \Psi_\lambda(C^1_{b,2}(E))$.

Remark: $\mathcal{D}(L; C^1_{b,2}(E)) \subset \mathcal{D}(L; UC_{b,2}(E))$

Key Proposition

(i) $\mathcal{FC}_b^\infty \subset \mathcal{D}(L; C^1_{b,2}(E)) \subset \mathrm{Dom}(\overline{\mathcal{L}_2})$

(ii) For $F \in \mathcal{FC}_b^\infty$,

$$LF(w) = \frac{1}{2}\mathrm{Tr}(D_H^2 F(w)) + \frac{1}{2}\langle w, (\Delta_x - \kappa) D_H F(w(\cdot))\rangle, w \in E.$$

(iii) For $F \in \mathcal{D}(L; C^1_{b,2}(E))$,

$$\overline{\mathcal{L}_2} F = LF + (b(\cdot), DF)_E,$$

where $b : \mathrm{Dom}(b) \subset E \to E$ is a measurable mapping with $\mathrm{Dom}(b) = \mathcal{C}$ defined by

$$b(w)(\cdot) := \frac{1}{2}(\kappa w(\cdot) - \nabla U(w(\cdot))).$$

Hence to solve the equation ($\sharp$), it is sufficient to show that our candidate

$$\Phi = \int_0^\infty e^{-\lambda t} P_t F dt, \ \lambda > \frac{K_1}{2} + r^2,$$

satisfies

(i)	$\Phi \in \mathcal{D}(L; C^1_{b,2}(E))$,
(ii)	$\lambda\Phi - L\Phi - (b(\cdot), D\Phi)_E = F$

$\Longrightarrow$ How to check the assertions (i) and (ii)?

Fact (E. Priola, '98)

$F \in \mathcal{D}(L; C^1_{b,2}(E))$ is equivalent to

(i-1): $\sup_{t>0} \frac{1}{t} \|R_t F - F\|_{b,2} < \infty$

(i-2): $\exists G (= LF) \in C^1_{b,2}(E)$ s.t.

$$\lim_{t \to 0} \frac{1}{t}(R_t F(w) - F(w)) = G(w), \ w \in E.$$

$\Longrightarrow$ To show the item (i-2), we transform $\frac{1}{t}(R_t \Phi(w) - \Phi(w))$ as follows:

- $S(b)_t := \int_0^t S_{t-s} b(X_s^w) ds$

$$
\begin{aligned}
&\frac{1}{t}\left(R_t\Phi(w)-\Phi(w)\right)\\
&= \frac{1}{t}\mathbb{E}\left[\Phi(Y_t^w)-\Phi(w)\right]\\
&= \frac{1}{t}\mathbb{E}\Big[\Phi\big(X_t^w - S(b)_t\big)-\Phi(w)\Big]\\
&= \frac{1}{t}\mathbb{E}\left[\Phi(X_t^w)-\Phi(w)\right]\\
&\quad -\mathbb{E}\Big[\int_0^1 \Big(D\Phi\big(X_t^w-\theta S(b)_t\big), \frac{1}{t}S(b)_t\Big)_E d\theta\Big]\\
&= \frac{1}{t}\left(P_t\Phi(w)-\Phi(w)\right)\\
&\quad -\int_0^1 \mathbb{E}\Big[\Big(D\Phi\big(X_t^w-\theta S(b)_t\big), \frac{1}{t}S(b)_t\Big)_E\Big]d\theta
\end{aligned}
$$

By letting $t \searrow 0$ on the right-hand side, we have

- $$\frac{1}{t}\big(P_t\Phi(w) - \Phi(w)\big) = \frac{e^{\lambda t}-1}{t}\int_t^\infty e^{-\lambda s}P_sF(w)ds - \frac{1}{t}\int_0^t e^{-\lambda s}P_sF(w)ds \longrightarrow \lambda\Phi(w) - F(w)$$

- $$\int_0^1 \mathbb{E}\Big[\Big(D\Phi(X_t^w - \theta S(b)_t), \frac{1}{t}S(b)_t\Big)_E\Big]d\theta \longrightarrow \int_0^1 \mathbb{E}\Big[\big(D\Phi(X_0^w), b(X_0^w)\big)_E\Big]d\theta = \big(D\Phi(w), b(w)\big)_E$$

Hence it holds that

$$\lim_{t\to 0}\frac{1}{t}(R_t\Phi(w)-\Phi(w))$$

$$= \quad \lambda\Phi(w)-F(w)+\big(D\Phi(w),b(w)\big)_E$$

$$\Downarrow$$

To show (RHS)$\in C^1_{b,2}$, we need some regularities of the function $D\Phi$.

♣ Representation formula (estimate) for the gradient of $P_t \implies \Phi\in C_b^2(E)$

(K.: Bull. Sci. Math. (2004), Potential Anal. (2005)

$\leftarrow$ from the view point of stochastic flow)

$$\|D(P_tF)(w)\|_E\le e^{(\frac{K_1}{2}+r^2)t}P_t(\|DF\|_E)(w)$$

The gradient estimate leads us to the estimate

$$\begin{aligned}
\|D\Phi\|_\infty &\le \int_0^\infty e^{-\lambda t}\|D(P_tF)\|_\infty dt \\
&\le \|DF\|_\infty \int_0^\infty e^{(\frac{K_1}{2}+r^2-\lambda)t} dt \\
&< \infty \quad \text{under } \lambda > \frac{K_1}{2} + r^2
\end{aligned}$$

By these arguments, we have shown the assertions (i) and (ii) !

REMARK: Above proof is not complete!

- $P_tF(w)$ and $P_t\Phi(w)$ are defined only on $\mathcal{C}$.
- To show $\Phi \in C_b^2(E)$, we need $U \in C^2(\mathbb{R}^d, \mathbb{R})$.

$$\Downarrow$$

To give a complete proof, we should introduce approximation functions for the drift b.

- Yosida approximation $\longrightarrow$ Lipschitz continuity
- Mollifization technique on infinite dimensions

$$b_\beta(w) := \int_E e^{\beta B} b(e^{\beta B}w + y) N_{\frac{1}{2}B^{-1}(e^{2\beta B}-1)}(dy)$$

(B: negative definite self-adjoint op with B^{-1} is of trace class)

Further Problems:

- Gibbs measures on $C(\mathbb{R}, \mathbb{R}^d)$ with two-body potentials (Osada-Spohn, Funaki, Hariya, etc)

$$\tilde{H}(w) = H(w) + \iint_{\mathbb{R}^2} W(x-y, w(x)-w(y))\,dx\,dy$$

- $P(\phi)_2$-time evolution ??