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Jump processes of mixed order
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1 Review.

1.1 Harmonic inequalities.

Connections between

(i) self adjoint operator L
(ii) Dirichlet form $(\mathcal{E}, \mathcal{F})$
(iii) Markov process $(X_t,\ t \geqq 0)$

Harmonic functions $Lh(x) = 0$, $x \in D$.
Heat equation $\dot{u} = Lu$ (HE).
If X has transition density $p(t, x, y)$, $x, y \in \mathbb{R}^d$; $P^x(X_t \in B) = \int_B p(t, x, y)dy$, then $\dot{p} = Lp$.

Question: What kind of regularity properties does one have for the harmonic functions and the solutions of (HE)?

Example (historically very important): $L = \sum_i \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}) = \nabla \cdot (a\nabla)$;

$a_{ij}(x) = a_{ji}(x)$: measurable in x.
Uniformlly elliptic: $0 < \lambda_1|\xi|^2 \leqq \xi^T a(x)\xi \leqq \lambda_2|\xi|^2$, $\forall \xi \in \mathbb{R}^d$.
Energy integral: For $f, g \in C_0^\infty$ $(-Lf, g) = \int -\nabla \cdot (a\nabla f)g = \int (\nabla g)^T a\nabla f = \mathcal{E}(f, g)$.
If $a_{ij}(x)$ is C^2 in x, standard methods (Schauder estimates, etc.) give continuity of solns of (HE).
Note, without differentiability of a_{ij} we can still make sence of (HE) by considering weak solutions. Then what if $a_{ij}(x)$ just measurable in x?
This problem was solved by de Giorgi, Moser, Nash (1959–61). (Fields-prize-worth accomplishment, but since 3 people solved it, it wasn't aworded. Nash later won Nobel prize in economy.)
Moser (1961): Elliptic Harnack inequality for $\nabla \cdot (a\nabla)$ (as a tool to solve the problem).

Why important? For $F : \mathbb{R} \to [1, 2] \in C^\infty$, consider a non linear PDE $\nabla \cdot (F(u)\nabla u) = 0$.
Suppose we knew the solution u and if we put $a = F(u)$, then $\nabla \cdot (a\nabla u) = 0$. If you know that a is continuous, then a general theory on harmonic equation implies that u exists and is continuous, and we have a consistent solution. Since we don't know u in the beginning, it was important to start just with assumption that a is measurable.

Definition. Elliptic Harnack inequality (EHI): Harnack inequality for $Lh = 0$ (Simpler to explain than Harnack inequality for heat equation):
L (or $(\mathcal{E}, \mathcal{F})$ or X) satisfies EHI if $\exists C_H < \infty$, s.t., if $B = Ball(x_0, R)$ and h is harmonic in B with $h \geqq 0$, then

$$\sup_{B(x_0,R/2)} h \leqq C_H \inf_{B(x_0,R/2)} h. \qquad \diamond$$

Example: $h(x) = P^x(X_\tau \in \Gamma)$.
What EHI says: Mixing goes on inside a ball.
There are many applications, e.g., Strook pushed Nash's results from probability setting.

1.2 Jump process.

(i) Arise naturally in probability theory.
$H \subset \mathbb{R}^d$: hyperplane. Diffusion in $\mathbb{R}^d$ restricted on hyperplane H, is a jump process.
(ii) Arises sometimes in analysis ('Trace of diffusion').
(iii) Barlow – Bass – Gui (jump process related to a certain non-linear PDE)

Fundamental object — jump rate $x \to y$: $n(x,y) = n(y,x)$, $x, y \in \mathbb{R}^d$.
Energy form: $\mathcal{E} = \mathcal{E}[n]$;
$\mathcal{E}[n](f,f) = \int_{\mathbb{R}^d} (f(x) - f(y))^2 n(x,y)\, dx\, dy$, $x \in L^2$. (Maybe ∞.)
Necessary for a 'nice' process:
$N_1(x) := \int_{B(x,1)^c} n(x,y)\, dy \in L^1_{loc}$
(so that the process does not make too many big jumps).
$M_1(x) := \int_{B(x,1)^c} |x-y|^2 n(x,y)\, dy \in L^1_{loc}$.
Note that these imply that if $f \in C^1_c(\mathbb{R}^d)$, then $\mathcal{E}[n](f,f) < \infty$.
$Lf(x) = p.v. \int (f(y) - f(x)) n(x,y)\, dx\, dy$, $f \in C^2_c(\mathbb{R}^d)$
Simplest example: stable process with index α ($\in (0,2)$).
$n(x,y) = |x-y|^{-d-\alpha}$: Lévy process (translation invariant).
Properties:

(i) $E^0[e^{\sqrt{-1}\lambda X_t}] = e^{-t|\lambda|^\alpha}$. ($\alpha = 2$ is BM.)
(ii) For $d = 1$, X point recurrent $\Leftrightarrow \alpha \in (1,2)$.

Long studied since 1930s.

Perturabation of stabled process. Study started only for a few years (Bass–Leven, Kumagai–Chen, on general metric spaces).
$c_1|x-y|^{-d-\alpha} \leqq n(x,y) = n(y,x) \leqq c_2|x-y|^{-d-\alpha}$
(the bounds which we will below write $n \asymp |x-y|^{-d-\alpha}$).
May not be continuous, and may not be translationally invariant (Fourier transform may not work!).
Results:

(i) $\exists$ strong Markov (Feller) process Y associated with Dirichlet form $(\mathcal{E}[n], \mathcal{F}[n])$
(ii) Y has a transition density $p(t,x,y) \asymp t^{-d/\alpha} \wedge t|x-y|^{-d-\alpha}$
(iii) EHI holds for Y
(iv) Harmonic functions are continuous

2 Problem and results.

Want to study $n(x,y)$ satisfying
(A1) $K_1|x-y|^{-d-\alpha} \leqq n(x,y) \leqq K_2|x-y|^{-d-\beta}$, $|x-y| < 1$, for $K_1 > 0$, $K_2 > 0$, $0 < \alpha < \beta < 2$.
(A2) $n(x,y) = 0$ if $|x-y| > 1$ (could be relaxed).
(Motivation is to generalize previous, satisfactory results; not from specific examples in mind.)

Questions:

(i) Existence of strong Markov process associated with $\mathcal{E}[n]$
(ii) Continuity of harmonic functions
(iii) Harnack inequalities, etc.

Given regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, general theory gives Hunt processes (in particular, strong Markov).

But: a difficulty. A natural thing is to look at $(\mathcal{E}[n], \mathcal{F}[n])$, which turned out to be hard to prove regularity.

Want: $\forall f \in \mathcal{F}[n] \exists g_n \in \mathcal{F}[n] \cap C(\mathbb{R}^n)$; $\mathcal{E}_1(f - g_n) = \mathcal{E}(f - g_n) + \|f - g\|_2^2 \to 0$

Straightforward estimates doesn't give this. Instead: look at $(\mathcal{E}[n], \mathcal{F}')$ where $\mathcal{F}' = \bar{C_c^1(\mathbb{R}^d)}^{\mathcal{E}_1}$ (We know that C_c^1 is of finite energy, so this is regular by definition.)

Penalty for cheating:

(i) It may be hard to prove that given f is in $\mathcal{F}'$.
(ii) Need to prove process Y associated with $(\mathcal{E}, \mathcal{F}')$ is conservative (i.e., do not go to infinity in finite time).

Results.

Theorem 1.

(i) $\exists$ Hunt process Y on $S = \mathbb{R}^d \setminus N$ (N a null set) associated with $(\mathcal{E}, \mathcal{F}')$.
(ii) Y has transition density $p(t, x, y)$ with $p(t, x, y) \leqq c_1 t^{-\alpha/d}$.
(iii) $\exists t_0 = t_0(\alpha, \beta, \kappa_1, \kappa_2)$; $P^x[\sup_{s \leqq t_0} |Y_s - x| \geqq 1/4] \leqq 1/4$.
(This and strong Markov imply the next:)
(iv) Y is conservative
(v) Let $B = B(x_0, 2)$, Y^B be Y killed on exiting B, and $P^B(t, x, y)$ be transition density of Y^B. Then $P^B(t, x, y) \geqq C_1, x, y \in B(x_0, 1)$, $1/2 \leqq t \leqq 2$.
(This means that there are no sets which the process avoids; minimal regularity.) ◇

Theorem 2. EHI holds for $B(x_0, 2)$. ◇

Proofs use Nash – Stroock ideas.

Theorem 3. $\exists n(x, y)$ satisfying (A1), (A2) and harmonic $h(x)$ s.t., h is not uniformly continuous. ◇

Example of h in Theorem 3 for $d = 2$: Starting with a Lévy process $n_0(x, y) = n_0(0, y - x)$ and constants $0 < a_1 < a_2 < 2$, let $m(x_1, x_2) = n_0(0, (x_1, x_2)) = |x_1|^{-a_1 - 2} \wedge |x_2|^{-a_2 - 2}$, $|x_1|, |x_2| \leqq 1$, and $m(x_1, x_2) = 0$, otherwise.

Let $Y_t = (Y_t^1, Y_t^2)$ be its Levy process. Projection of Levy is Levy, so Y_t^1 and Y_t^2 are 1dim Levy (not independent), whose jump measures are given by integration:

$n_1(x_1) = \int_{-1}^{1} m_1(x_1, x_2) dx_2 = -c_1 + c_2 |x_1|^{-1-\alpha_1}$

and

$n_2(x_2) = \int_{-1}^{1} m_1(x_1, x_2) dx_1 = -c_3 + c_4 |x_1|^{-1-\alpha_2}$.

Y^1 and Y^2 are essentially (locally) stable with indices α_1, α_2, where $\alpha_1 = (a_1 + 1)(a_2 + 1)/(a_2 + 2) - 1$.

Lemma 4.

(i) n_0 satisfies (A1) with $\alpha = \alpha_1$, $\beta = \alpha_2$.
(ii) We can choose a_1, a_2 s.t., $0 < \alpha_1 < \alpha_2 < 1$. ◇

Set $V = \{(x_1, x_2) \mid |x_1| < |x_2|\}$ and let $Y_0 = 0$.

Z_t: 1-dim stable $\alpha \in (0, 1)$. Then

$\lim_{t \downarrow 0} |Z_t| t^{-1/\alpha - \epsilon} = \infty$, a.s.

$\lim_{t \downarrow 0} |Z_t| t^{-1/\alpha+\epsilon} = 0$, a.s.

So, $\exists \delta(w)$; $(\forall 0 < t < \delta(w))$

$0 < |Y_t^1| < t^{1/\alpha_1 - eps} < t^{1/\alpha_2 + eps} < |Y_t^2|$.

So for short time the process Y is in the cone V.

Now define $n(x,y) = m(|x_1 - y_1|, |x_2 - y_2|)$ if $x, y \in V, |x-y| < 1$,

and $n(x,y) = m(|x_2 - y_2|, |x_1 - y_1|)$ if $x, y \in V^c$,

and $n(x,y) = |x_1 - y_1|^{-2-a_1} \wedge |x_2 - y_2|^{-2-a_2}$ if $x_1 \in V, y \in V^c$ or $y \in V, x \in V^c$

and $n(x,y) = 0$ if $|x-y| \geqq 1$.

X: Corresponding process.

Behavior of X:

If $X_0 = x \in V$ close to 0, X stays in V for a positive time.

If $X_0 = x' \in V^c$ then X stays in V^c for a positive time. (In fact it moves somewhat along positive x_1 axis for some time.)

Put $D = (-\eta, \eta)^2$, $f(x) = 1_{D \cap V}$, and $h(x) = E^x f(X_{\tau_D})$. Then h is not continuous at 0 (because it is the probability that X exists the box from y direction).