Stochastic PDEs driven by space-time white noise with two reflecting walls and related problems

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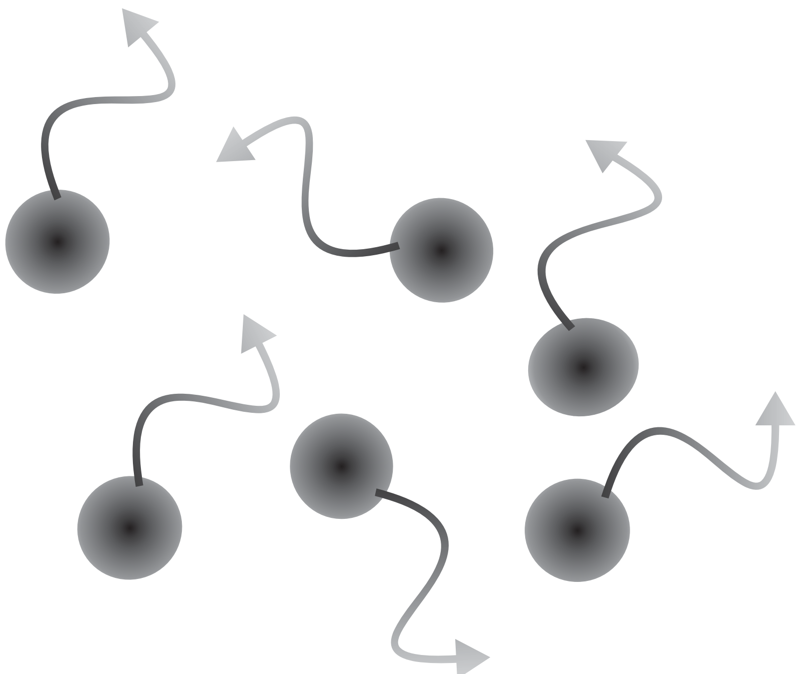
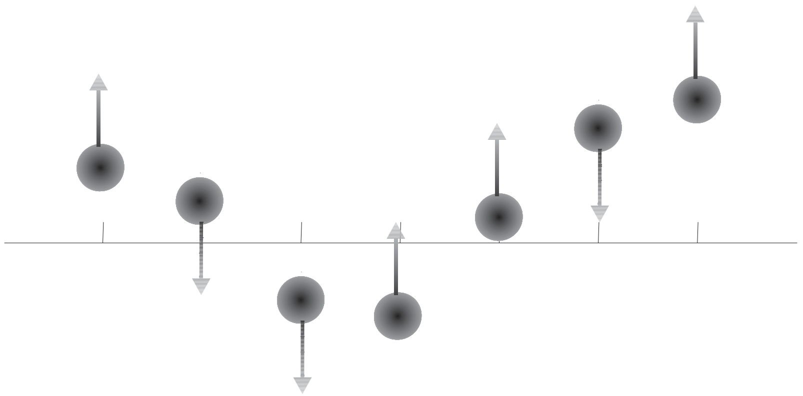
Microscopic phenomena

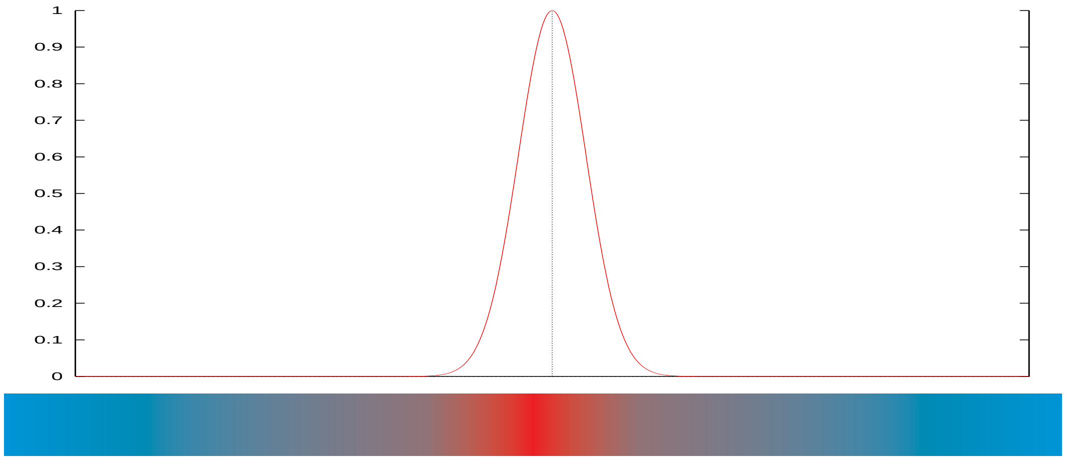


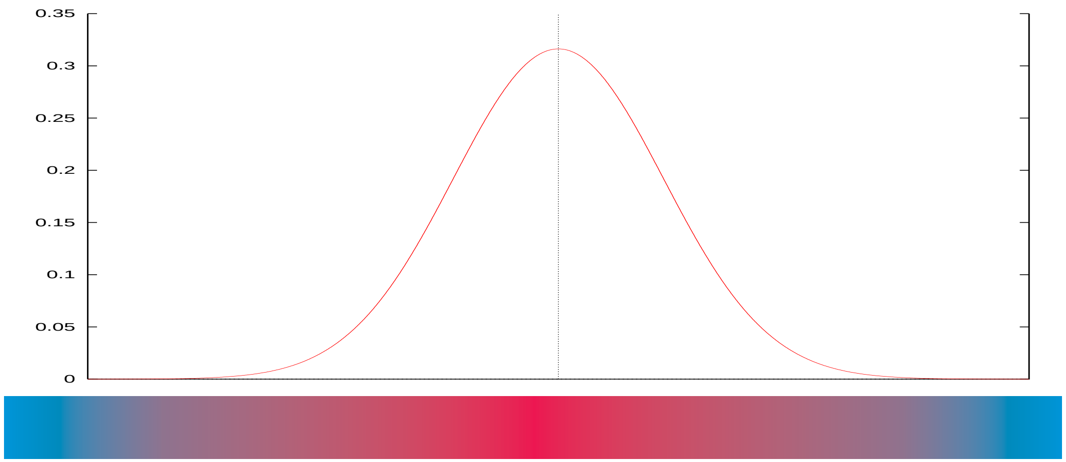


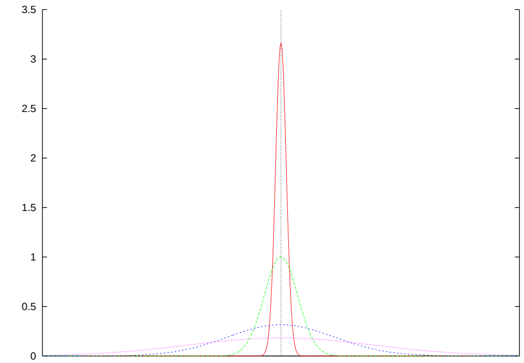
☞ Each particle moves *randomly*.

☞ There are some *interactions* among particles.

Macroscopic phenomena







diffusive phenomena

- ☞ We observe *smooth* time evolution.
- ☞ or they are *stationary*.

discrete free interface model (static)

The interface is a function $\varphi : \mathbb{Z}^d \to \mathbb{R}$ (a "graph" of $\mathbb{R}$-valued function), of which *energy* on compact $\Lambda \subset \mathbb{Z}^d$ is given by the Hamiltonian

$$H_\Lambda(\varphi) = \frac{1}{2} \sum_{i,j \in \Lambda} V(\varphi(j) - \varphi(i)) + \sum_{i \in \Lambda, j \notin \Lambda} V(\varphi(j) - \varphi(i)),$$

with $V : \mathbb{R} \to \mathbb{R}$ even, uniformly convex, $V(0) = 0$.

The statistical properties of the interface are described by *Gibbs measure*, a probability measure $P_\Lambda^{\psi,\beta}$ on $\mathbb{R}^{\mathbb{Z}^d} = \{\varphi : \mathbb{Z}^d \to \mathbb{R}\}$

$$P_\Lambda^{\psi,\beta}(d\varphi) = (Z_\Lambda^{\psi,\beta})^{-1} \exp\{-\beta H_\Lambda(\varphi)\} \prod_{i \in \Lambda} d\varphi_i \prod_{j \notin \Lambda} \delta_{\psi(j)}(d\varphi_j)$$

dynamical point of view

"*Gibbs states become more interesting when they are viewed as the equilibrium state of a dynamical system and, in addition, the dynamics often provides a natural approach to the analysis of Gibbs state.*"—D.W. Stroock, *Logarithmic Sobolev Inequalities for Gibbs States*, LNM 1563, pp. 194–228 (1993).

- ☞ There may be several dynamical systems which posesses the Gibbs state as equilibrium.
- ☞ In this talk, we investigate a continuous interface on one-dimensional continuum fields.

$$\text{Interface} = \varphi : \mathbb{R} \to \mathbb{R}.$$

- ☞ We will choose stochastic PDEs as a dynamic model (very natural, I believe!).

references

☞ G. Giacomin, *Aspects of statistical mechanics of random surfaces*, Note of lectures given at IHP, fall 2001, http://www.proba.jussieu.fr/pageperso/giacomin/pub/publicat.html

☞ Y. Velenik, *Localization and delocalization of random interfaces*, Note of lectures given at Leipzig, fall 2005, http://www.univ-rouen.fr/LMRS/Persopage/Velenik/research.html

☞ T. Funaki, Lectures on probability theory and statistics, LNM 1869. (http://www.ms.u-tokyo.ac.jp/~funaki/)

linear SDE

$$dX(t) = A(t)X(t)dt + \sigma(t)dW(t).$$

The solution is

$$X(t) = \Phi(t)\left(X(0) + \int_0^t \Phi(s)^{-1}\sigma(s)dW(s)\right).$$

$\Phi(t)$ solves $d\Phi(t) = A(t)\Phi(t)dt$, $\Phi(0) = \mathrm{Id}$.

☞ $m(t) := E[X(t)]$.

☞ $\rho(s,t) := E[(X(s) - m(s)) \otimes (X(t) - m(t))]$.

☞ $V(t) := \rho(t,t)$.

$$\begin{aligned} \dot{m}(t) &= A(t)m(t) \\ \dot{V}(t) &= A(t)V(t) + \sigma(t)\sigma(t)^\dagger + V(t)^\dagger A(t)^\dagger \end{aligned} \qquad (1)$$

stationary solutions to SDEs

Assume $A(t) \equiv A$, $\sigma(t) \equiv \sigma$. $(\Phi(t) = \exp\{tA\})$.

equilibrium $\Rightarrow V(t) \equiv$ Const.

$$V(t) = e^{tA}V(0)e^{tA^\dagger} + \int_0^t e^{sA}\sigma\sigma^\dagger e^{sA^\dagger}\, ds$$

It is needed

$$e^{tA}V(0)e^{tA^\dagger} = \int_t^\infty e^{sA}\sigma\sigma^\dagger e^{sA^\dagger}\, ds$$

$$\Rightarrow V(0) = \int_0^\infty e^{sA}\sigma\sigma^\dagger e^{sA^\dagger}\, ds$$

$$\Rightarrow V(t) \equiv V, \quad AV + \sigma\sigma^\dagger + VA^\dagger = 0$$

☞ A must be negative definite (all e.v.'s of $A < 0$.).

distribution of Langevin's equation

Let us consider the following SDE.

$$\begin{cases} dX(t) = AX(t)\,dt + \sigma\,dW(t) \\ X(0) = \xi \end{cases}$$

Then we have Mehler's formula:

$$P(X(t) \in dx) = \frac{1}{\sqrt{(2\pi)^d |V|}} \exp\left\{ -\frac{1}{2} \left\langle x - m(t), V(t)^{-1}(x - m(t)) \right\rangle \right\},$$

where $V(t)$ is given by (1).

equilibrium of Langevin's equation

Suppose that all the eigen values of A have negative real parts and ξ is a Gaussian random variable with zero-mean and covariance $V = \int_0^\infty e^{sA}\sigma\sigma^\dagger e^{sA^\dagger}\, ds$. Then $X(t)$ is a stationary, zero-mean Gaussian process of which covariance function is given by

$$\rho(s,t) = \begin{cases} e^{(s-t)A}V, & 0 \le t \le s < \infty \\ Ve^{(t-s)A^\dagger}, & 0 \le s \le t < \infty. \end{cases}$$

- ☞ linear case: all quantity are computable!
- ☞ General definitions of equilibrium?

Invariant measures

Let $X(t)$ be a S-valued process on $(\Omega, \mathscr{F}, P)$.

A probability measure μ on $(S, \mathscr{S})$ is called invariant if $X(0)$ is μ-distributed ($P(X(0))^{-1} = \mu$) then $X(t)$ is also μ-distributed, that is,

$$P_\mu(X(t) \in A) = \mu(A) \quad (= P_\mu(X(0) \in A)).$$

☞ P_x: prob. meas on $(\Omega, \mathscr{F})$ s.t. $P_x(X(0) = x) = 1$.

☞ $P_\mu(A) := \int_S P_x(A)\, \mu(dx)$, i.e., a probability (law) on $(\Omega, \mathscr{F})$ that the Markov process $X(t)$ has initial distribution μ.

Usuful formulations

$P_x(x(t) \in A) = E_x[1_A(X(t))]$ leads us ...

A prob. meas. μ on $(S, \mathscr{S})$ is invariant

$$\iff \int_S E_x[F(X(t))]\,\mu(dx) = \int_S F(x)\,\mu(dx).$$

Similary, we may formulate

A prob. meas. μ on $(S, \mathscr{S})$ is invariant

$$\iff E_\mu[F(X(t))] = E_\mu[F(X(0))].$$

The *test function* F may be taken from a measure determining family of $(S, \mathscr{S})$ ($C_b(S)$ etc.).

reversible measure

Let $X(t)$ be a S-valued process on $(\Omega, \mathscr{F}, P)$.

A probability measure μ on $(S, \mathscr{S})$ is called reversible if $X(0)$ is μ-distributed $(P(X(0))^{-1} = \mu)$ then

$$X(0) \in A \longrightarrow X(t) \in B$$

$$X(t) \in A \longrightarrow X(0) \in B$$

occurs in the same probability for every $t > 0$, that is,

$$P_\mu(X(0) \in A \wedge X(t) \in B) = P_\mu(X(0) \in B \wedge X(t) \in A).$$

The reversible measure μ is also an invariant measure for $X(t)$. We can reformulate the above by

$$\int_A P_x(X(t) \in B)\, \mu(dx) = \int_B P_x(X(t) \in A)\, \mu(dx).$$

Usuful formulations

In a similar manner to the case of invariant measures. . .

A prob. meas. μ on $(S, \mathscr{S})$ is reversible $\iff$

$$\int_S F(x) E_x[G(X(t))]\, \mu(dx) = \int_S G(x) E_x[F(X(t))]\, \mu(dx).$$

Similary, we may formulate

A prob. meas. μ on $(S, \mathscr{S})$ is reversible $\iff$

$$E_\mu[F(X(0))G(X(t))] = E_\mu[G(X(0))F(X(t))].$$

The *test function* F may be taken from a measure determining family of $(S, \mathscr{S})$ $(C_b(S)$ etc.).

Langevin's equation again

$dX(t) = \frac{1}{2}AX(t)\,dt + dB(t)$, A: negative definite. Suppose also that A is symmetric.

This time, $AV = VA$ and $2A = -V^{-1}$ for $V = \int_0^\infty e^{2sA}\,ds$.

The reversible measure μ on $\mathbb{R}^d$ for Langevin's dynamics is given by

$$\mu(dx) = \frac{1}{Z}\exp\left\{\frac{1}{2}\langle Ax, x\rangle\right\}dx.$$

perturbations of Langevin's Dynamics

$$dX(t) = \frac{1}{2}(AX(t) - \nabla U(X(t)))\,dt + dB(t).$$

To avoid the difficulty from the integrability, assume U is bounded with bounded derivatives.

☞ $dY(t) = \frac{1}{2}AY(t)\,dt + dB(t)$.
Then the law Q of Y on $C([0,T],\mathbb{R}^d)$ is given by Cameron–Martin–Maruyama–Girsanov.

☞ The law R of X is also concretely given.

☞ Easily compute dR/dQ.

☞ Using Itô formula allows us to escape the stochastic integral.

The reversible measure μ for $X(t)$ is given by

$$\mu(dx) = \frac{1}{Z}\exp\left\{-U(x) + \frac{1}{2}\langle Ax, x\rangle\right\}dx.$$

summary of reversible measures

Note that $\nabla(\frac{1}{2}\langle Ax, x\rangle) = Ax$.

Let $V : \mathbb{R}^d \to \mathbb{R}$ given (called, potential, energy, or *Hamiltonian*). A reversible measure of the dynamics obeying the following stochastic ordinary differential equation

$$dX(t) = -\frac{1}{2}\nabla V(X(t))\,dt + dB(t)$$

is given by the following (Gibbs type) formula:

$$\mu(dx) = \frac{1}{Z}\exp\{-V(x)\}\,dx.$$

Z is the normalizing constant (making μ probability measure) and is sometimes called a *partition function*.

However, is the reversible measure unique?

Analytic quantity

We fix an SDE: $dX(t) = -\frac{1}{2}\nabla V(X(t))\,dt + dB(t)$. The generator of X is defined by

$$L := \frac{1}{2}\Delta - \frac{1}{2}\nabla V \cdot \nabla,$$

namely

$$(Lf)(x) = \frac{1}{2}(\Delta f)(x) - \frac{1}{2}\langle \nabla V(x), \nabla f(x) \rangle .$$

Let $\mu(dx) = Z^{-1} \exp\{-V(x)\}\,dx$ be a reversible measure of X. Define a *bilinear form* $\mathscr{E}$ by

$$\mathscr{E}(f,g) := \frac{1}{2}\int_{\mathbb{R}^d} \langle \nabla f, \nabla g \rangle\, \mu(dx).$$

for nice functions $f, g : \mathbb{R}^d \to \mathbb{R}$.

integration by parts w.r.t μ

Suppose $\lim_{|x|\to\infty} V(x) = +\infty$. Straightforward computation leads us to,

$$\frac{1}{2}\int_{\mathbb{R}^d} \langle \nabla f, \nabla g\rangle \, \mu(dx) = -\int_{\mathbb{R}^d} (Lf)(x) g(x) \, \mu(dx)$$
$$= -\int_{\mathbb{R}^d} (Lg)(x) f(x) \, \mu(dx).$$

☞ under μ, L can be considered as a usual second order differential operator.

☞ L makes it possible to execute the calculus on a Gibbs state μ.

entropy

μ, ν: two probability measures on $\mathbb{R}^d$.

Define a relative entropy of μ with respect to ν by

$$H(\mu|\nu) := \begin{cases} \int_{\mathbb{R}^d} \frac{d\mu}{d\nu} \log \frac{d\mu}{d\nu} \nu(dx) & \mu \ll \nu \\ \infty & \text{otherwise.} \end{cases}$$

If $\mu(dx) = f(x)\,\nu(dx)$, $H(\mu|\nu) = \int f(x) \log f(x)\,\nu(dx)$.

Somebody may assert that this H must be called "*negative*" entropy!

log-Sobolev inequality

If $\mu(dx) = f(x)\,\nu(dx)$, we have $H(\mu|\nu) \le C\mathscr{E}(\sqrt{f}, \sqrt{f})$, namely,

$$\int_{\mathbb{R}^d} f(x) \log f(x)\,\nu(dx) \le C \int_{\mathbb{R}^d} \left\langle \nabla\sqrt{f(x)}, \nabla\sqrt{f(x)} \right\rangle \nu(dx).$$

☞ define $(P_t f)(x) := E_x[f(X(t))]$ and $g_t(x) := P_t f(x)$.

☞ check that $\int f(x) \log f(x) \nu(dx) = -\int_0^\infty \frac{d}{dt} \int g_t(x) \log g_t(x)\,\nu(dx) dt$.

☞ note that $\frac{d}{dt} P_t = L P_t$.

It is easy to see

$$\|\nu - \mu\|^2_{\text{total var}} \le 2H(\mu|\nu).$$

convergence of dynamics

$\mu_t(dx)$: distribution of $X(t)$ on $\mathbb{R}^d$.

Assume $\mu_t(dx) = f_t(x)\mu(dx)$ and $H(\mu_0|\mu) < \infty$.

Assume moreover that $\mu_t(dx)$ is absolutely continuous with respect to dx. Then we have

$$\frac{d}{dt}H(\mu_t|\mu) = -4\mathscr{E}(\sqrt{f_t}, \sqrt{f_t}).$$

Combining with log-Sobolev inequality, we have

$$H(\mu_t|\mu) \le e^{-4t/C}H(\mu_0|\mu).$$

That is, the law of $X(t)$ converges to the reversible distribution exponentially fast.

Infinite dimensional case

Define a linear operator $Af(x) = \frac{d^2}{dx^2}f(x)$ on $L^2(0,1)$ with a domain the completion of $D(A) := \{f \in C^2(0,1); f(0) = f(1) = 0\}$.

Then the eigen space of A is clearly $\{\sin n\pi x\}_{n=1}^{\infty}$ and the eigen values are $\{-n^2\pi\}$, namely A is a strictly negative definite (unbounded) operator.

☞ If A is considered with Neumann conditions ("1" is an eigen function), or over $\mathbb{R}$, A is NOT negative.

☞ Today I always assume A is considered with Dirichlet conditions. Such a twice differential operator will be simply denoted by Δ. (Δ always denotes the closed Laplacian with Dirichlet boundary conditions).

Stochastic Partial Differential Equations

Let us consider the follwing SDE:

$$dX(t) = \frac{1}{2}\Delta X(t)\,dt + dB(t).$$

This may be called a "stochastic partial differential equation".

☞ Δ is unbounded (non continuous). The Itô formula may fail. How to handle such an operator?

☞ $B(t)$ is "L^2-valued"-Brownian motion? Does it mean Gaussian distributed on L^2? In infinite dimension, we need to pay attentions to handle such measures.

If we were able to reach the "solution", it may have a reversible measure μ "*defined*" by

$$\mu(dw) = \frac{1}{Z}\exp\left\{\frac{1}{2}\langle \Delta w, w\rangle\right\}dw,$$

where dw denotes the Feynman measure, possibly infinite dimensional analogue of Lebesgue measure.

Gaussian measure on a Banach space B

We call μ, a probability measure on B, Gaussian if for every $\phi \in B^*$, considered as a random variable on $(B, \mathscr{B}(B), \mu)$, the law of ϕ is Gaussian measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

A linear subspace $H \subset B$, with Hilbert norm $|\cdot|_H$, is said to be a *reproducing kernel space* for μ if H is complete, continuously embedded in B such that, for every $\phi \in B^*$, the law of ϕ is zero-mean Gaussian with covariance $|\phi|_H^2$.

For every symmetric Gaussian measure μ on B, there exists a unique reproducing kernel space H.

uniqueness of reproducing kernel space

Let E be another Banach space $E \hookrightarrow B$ continuously as a Borel set. If μ is a symmetric Gaussian on B and E, then the reproducing kernel space w.r.t. B and E are the same.

☞ B given

☞ μ given

☞ $H \hookrightarrow B$ uniquely determined.

e.g., $B := C([0,T], \mathbb{R}^d)$, μ : Wiener measure. Then H is so-called Cameron–Martin space,

$$H := \{f : [0,T] \to \mathbb{R}^d; \text{abs. conti.}, f'(t) \in L^2\}.$$

Abstract Wiener Space

B: separable Banach space.

H: Hilbert space ($H \hookrightarrow B$ conti., densely embedded).

μ: Gaussian measure on B such that

$$\int_B \exp\left\{\sqrt{-1}\,\langle w, \phi\rangle\right\}\mu(dw) = \exp\left\{-\frac{1}{2}|\phi|_H^2\right\}$$

for every $\phi \in B^* \subset H^* = H$.

The triplet (B, H, μ) is called an abstract Wiener space.

$\mu(\cdot - h)$ is equivalent to μ iff. $h \in H$, and

$$\frac{d\mu(\cdot - h)}{d\mu}(w) = \exp\left\{-\frac{1}{2}|h|_H^2 + \langle h, w\rangle\right\}.$$

H-derivative

A function $F : B \to \mathbb{R}$ is H-differentiable on $w \in B$ if there exists $DF(w) \in H$ such that

$$\frac{d}{dt}F(w+th)\Big|_{t=0} = \langle DF(w), h\rangle_H$$

is satisfied for every $h \in H$.

- ☞ $F(w+th) = F(w) + t\,\langle DF(w), h\rangle + o(|t|)$.
- ☞ If $F(w) = f(\langle w, \phi_1\rangle, \langle w, \phi_2\rangle, \ldots, \langle w, \phi_n\rangle)$, $\phi_i \in B^*$, then

$$DF(w) = \sum_{i=1}^{n} \partial_i f(\ldots)\phi_i(x) \in H.$$

Dirichlet form theory on B

Using H-derivative, we define a bilinear form

$$\mathscr{E}(F,G) := \frac{1}{2}\int_B \langle DF(w), DG(w)\rangle_H \, \mu(dw)$$

on an AWS (B,H,μ) with $D(\mathscr{E}) = \mathbb{D}_1^2(B)$. Then $(\mathscr{E}, D(\mathscr{E}))$ is a regular Dirichlet form (closed symmetric Markovian form). We can prove (S. Kusuoka, *Dirichlet forms and diffusion processes on Banach Space*, 82) that there exists a diffusion process on B which (weakly) solves to

$$dX(t) = -\frac{1}{2}X(t)\,dt + dB(t),$$

where $B(t)$ is B-valued Brownian motion.

However, our goal is still far away...

Gaussian measure on Hilbert space

B: separable Hilbert space.

$Q : B \to B$, strictly positive symmetric nuclear operator, $\operatorname{Ker} Q = \{0\}$

μ: Gaussian measure on B with covariance operator Q.

$H := Q^{1/2}(B)$, $\langle f, g\rangle_H := \left\langle Q^{-1/2} f, Q^{-1/2} g \right\rangle_B$.

$$\int_B \exp\left\{\sqrt{-1}\,\langle w, \phi\rangle_B\right\} \mu(dw) = \exp\left\{-\frac{1}{2}\langle Q\phi, \phi\rangle_B\right\}.$$

$$\mu(dw) = \frac{1}{Z}\exp\left\{-\frac{1}{2}\left\langle Q^{-1} w, w\right\rangle_B\right\} dw.$$

on B.

"computations" based on Feynman measure

$$\int_B \langle DF(w), DG(w)\rangle_H \, \mu(dw)$$

$$= \int_B \langle DF(w), DG(w)\rangle \, Z^{-1} \exp\left\{-\frac{1}{2}\left\langle Q^{-1}w, w\right\rangle_B\right\} dw$$

$$= \int_B \langle DF(w), DG(w)\rangle \, Z^{-1} \exp\left\{-\frac{1}{2}\langle w, w\rangle_H\right\} dw$$

$\longleftrightarrow dX(t) = dW(t) - \frac{1}{2}X(t)\,dt.$

Now, we shall take a concrete Hilbert space $B := H^{-1}(0,1)$, $H := L^2(0,1)$, and $Q := (-\Delta)^{-1}$.

Recall that we want to consider

$$\mu(dw) = \frac{1}{Z}\exp\left\{\frac{1}{2}\langle \Delta w, w\rangle\right\} dw,$$

on L^2.

SPDE on Feynman measure

Recall that (in finite dim)

$$\int_{H=B} \langle DF(w), DG(w)\rangle_H \, Z^{-1} \exp\left\{-\frac{1}{2}\langle(-\Delta)w, w\rangle_H\right\} dw$$

$$\longleftrightarrow dX(t) = \tfrac{1}{2}\Delta X(t)\,dt + dB(t).$$

$$= \int_B \langle DF(w), DG(w)\rangle_H \, Z^{-1} \exp\left\{-\frac{1}{2}\langle(-\Delta)(-\Delta)w, w\rangle_B\right\} dw$$

$$= \int_B \langle DF(w), DG(w)\rangle_H \, Z^{-1} \exp\left\{-\frac{1}{2}\langle(-\Delta)w, w\rangle_H\right\} dw$$

$$= \int_B \langle DF(w), DG(w)\rangle_H \, Z^{-1} \exp\left\{-\frac{1}{2}|w|^2_{H^1_0}\right\} dw$$

$$= \int_H \langle \nabla F(w), \nabla G(w)\rangle_H \, \beta(dw)$$

pinned Wiener measure and Laplacian

A Gaussian process (pinned B.m.) $X(t) \equiv X^{a \to b}(t)$,

$$dX(t) = dB(t) + \frac{b - X(t)}{1 - t} dt, \qquad X(0) = a$$

has a covariance function $\rho(s,t) = (s \wedge t) - st$, which verifies

$$((-\Delta)^{-1} f)(x) = \int_0^1 \rho(x,y) f(y)\, dy.$$

A pinned Wiener measure β, extended to $L^2(0,1)$, the law of $X^{0 \to 0}$ is a Gaussian measure on $L^2(0,1)$ with covariance operator $(-\Delta)^{-1}$.

white noise

Let us go to the *strong* formulation of SPDE.

The white noise on $[0, \infty)$ is a centered Gaussian random variable $W(t)$ with covariance

$$E[W(t)W(s)] = \delta_{t-s}.$$

It is easy to see that $W(t) = dB(t)/dt$ (Itô derivative, or Schwartz sense).

The space-time white noise is a centered Gaussian field on (x, t) with covariance $E[W(x,t)W(y,s)] = \delta_{x-y}\delta_{t-s}$.

- ☞ rigorous formulation is expected.
- ☞ prefer to fit to stochastic analysis.

simplest way

Let $(\mathscr{S}, \mathscr{B}(\mathscr{S}))$ be the Schwartz space and μ is a Gaussian measure on $(\mathscr{S}, \mathscr{B}(\mathscr{S}))$ with reproducing kernel space L^2. each element $w \in (\mathscr{S}, \mathscr{B}(\mathscr{S}), \mu)$ is called white noise.

- ☞ each stochastic quantity is a white noise functional.
- ☞ H. Holden, B. Oksendal, J. Uboe, T. Zhang, *Stochastic Partial Differential Equations*, Birkhauser, 1996.
- ☞ Itô calculus?

Easiest way—Brownian sheet approach

☞ T. Kitagawa, *Analysis of variance applied to function space*, Mem. Fac. Sci. Kyushu Univ. 6, 41–53, 1951

Let $E := [0,\infty)^2$, m: Lebesgue measure on E.

A random set function W on $\mathscr{B}(E)$ is called a white noise if

①$W(A) \sim \mathscr{N}(0, m(A))$

②$A \cap B = \emptyset \Rightarrow W(A)$ and $W(B)$ is independent and $W(A \cup B) = W(A) + W(B)$.

A process $\{B(x,t)\}_{(x,t)\in E}$ defined by $B(x,t) := W((0,x] \times (0,t])$ is called *Brownian sheet*.

properties of Brownian sheet

① $E[B(x,t)B(y,s)] = (x \wedge y)(t \wedge s)$.

② if x is fixed, $\{B(x,t)\}_{t \geq 0}$ is a Brownian motion.

③ $M(t) := B(t,t)$ is a martingale, of (non stationary) independent increments, and is not a Brownian motion.

define $\dot{B}(x,t) := \dfrac{\partial^2 B(x,t)}{\partial x \partial t}$ in the sense of Schwartz distribution, namely, for $\phi \in C_0^2(E)$,

$$\dot{B}(\phi) = \int_E B(x,t) \frac{\partial^2 \phi(x,t)}{\partial x \partial t}.$$

If we may "expect" the existence of the Itô integral,

$$\dot{B}(\phi) \text{must be} \iint \phi(x,t) W(dx, dt).$$

Itô integral with respect to Brownian sheet

Take $\phi(x,t) = 1_{[0,x]\times[0,t]}(x,t)$.

$$\dot{B}(\phi) = \int_0^t \int_0^x \frac{\partial^2 B(x,t)}{\partial x \partial t}(y,s)\, dy\, ds$$

$$= B(x,t) = \iint \phi(x) W(dw, dt).$$

It is certainly true; The theory of the Itô integral w.r.t. Brownian sheet can be constructed as a usual way:

☞ J. B. Walsh, *An Introduction to Stochastic Partial Differential Equations*, LNM1180, 265–439, 1984.

We denote the space-time white noise by $\dfrac{\partial^2 B(x,t)}{\partial x \partial t}$ or $B(dx,dt)$ as a formal Itô derivative.

cylindrical approach—easily handled

A stochastic process $W(t)$ is called a cylindrical Brownian motion on L^2 or *white noise process* if $W \equiv \{W(t;\psi)\}$ is a family of $\mathbb{R}$-valued stochastic process with a parameter family $\psi \in L^2$ such that

①$\forall\psi \in L^2$, $W(t;\psi)/\|\psi\|_{L^2}$ is a one-dimensional standard B.m.

②$\forall\alpha, \beta \in \mathbb{R},\ \psi, \varphi \in L^2$,

$$W(t;\alpha\varphi + \beta\psi) = \alpha W(t;\varphi) + \beta(t;\psi)$$

almost surely (for $\alpha, \beta, \varphi, \psi$).

☞ This time, the theory of the Itô integral is rather easy.

☞ spece-time white noise is a formal Itô derivative of $W(t)$.

Itô integral for cylindrical B.m.

$$\int_0^t \langle f(s), dW(s)\rangle_{L^2} := \sum_{k=1}^{\infty} \int_0^t \langle f(s), \psi_k\rangle \, dW(s;\psi_k).$$

For $\Phi(t) : L^2 \to H$, Hilbert–Schmidt s.t.

$$E\left[\int_0^T \|\Phi(t)\|_{\mathrm{HS}}^2 dt\right] < \infty,$$

We define the stochastic integral $\int_0^t \Phi(s) dW(s)$ by

$$\left\langle \int_0^t \Phi(s)\, dW(s), \phi \right\rangle_H = \int_0^t \langle \Phi(s)^* \phi, dW(s)\rangle_{L^2}\, ds, \quad \forall \phi \in H.$$

☞ $W(s)$ is not even an L^2-valued process, the Itô integral is actually H-valued process.

☞ see T. Funaki, *Stochastic Differential Equations*, Iwanami, 2005.

distribution valued process (really easiest)

Let (H^{-1}, L^2, μ) be an AWS.

A Wiener space associated to this abstract Wiener space is called white noise process.

① Actually it is the same with cylindrical approach.

② see also, G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge, 1992.

stochastic partial differential equations

We consider stochastic partial differential equations

$$dX(t) = \frac{1}{2}\Delta X(t)dt - F(t; X(t))dt + \sigma(X(t))\, dW(t),$$

with $W(t)$ is a white noise process ($dW(t)/dt$ is a space-time white noise), and Dirichlet boundary conditions on $(0,1)$.

Sometimes this equation is written in the following form:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - f(x,t; u(x,t)) + \sigma(x,t; u(x,t))\dot{W}(dx,dt).$$

Now we define its solution.

weak form solution

We call a function $X(t) \in L^2(0,1)$ is a solution to the SPDE if

$$\langle X(t),\phi\rangle - \langle X(0),\phi\rangle =$$

$$\int_0^t \left\langle X(s),\phi''\right\rangle\, ds - \int_0^t \langle F(X(s)),\phi\rangle\, ds + \int_0^t \langle \sigma(X(s))\phi, dW(s)\rangle$$

is satisfied for every $\phi \in C_0^2(0,1)$.

- ☞ Under Lipschitz conditions on the coefficients, we can prove the existence and uniqueness result.
- ☞ $X(t)$ actually stayes in $C([0,1])$ and $X(t)$ is $C([0,1])$-valued diffusion.
- ☞ The regularity is that $X(t)(x)$ is $(1/2-\varepsilon)$-Hölder in x and $(1/4-\varepsilon)$-Hölder in t.
- ☞ In higher space dimension, the solution does not stay in any function space.

mild solutions

Let $e^{t\Delta}$ be a semigroup. Then $X(t)$ is a solution to the SPDE if it satisfies

$$X(t) = e^{t\Delta}X(0) + \int_0^t e^{t\Delta} f(X(s))\, ds + \int_0^t e^{t\Delta}\sigma(X(s))dW(s).$$

- ☞ a weak form solution is also a mild solution.
- ☞ a mild form solution is also a weak form solution.

reversible measures

Let us consider an SPDE with an additive space-time white noise:

$$dX(t) = \frac{1}{2}(\Delta X(t) - V'(\cdot, X(t)))\,dt + dW(t).$$

Then the reversible measure μ for $X(t)$ is given by

$$\mu(dw) = \frac{1}{Z}\exp\left\{-\int_0^1 V(x, w(x))\,dx\right\}\beta(dw),$$

β is a Gaussian measure on $C([0,1])$ induced by a pinned Brownian bridge 0 to 0.

corresponding Dirichlet form

Define a bilinear form

$$\mathscr{E}(F,G) := \frac{1}{2}\int_{L^2} \langle \nabla F(w), \nabla G(w)\rangle_{L^2}\, \mu(dw)$$

for $F, G \in \mathscr{F}C_b^\infty$, where ∇ denotes the Fréchet derivative on L^2. Then the closure is a Dirichlet form corresponding to the SPDE (T. Funaki, *The reversible measures of multi-dimensional Ginzburg–Landau type continuum model*, Osaka J., 1991).

The Poincaré inequality and log-Sobolev inequality hold.